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ON EDGE WAVES IN STRATIFIED WATER ALONG A SLOPING BEACH

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Building on previous investigations, we show that Gerstner's famous deep water wave and the related edge wave propagating along a sloping beach, found within the context of water of constant density, can both be adapted to provide explicit free surface flows in incompressible fluids with arbitrary density stratification.

Keywords: Euler equations; free boundary; stratification; edge waves.

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1. Introduction

Stratification is ubiquitous in fluid motion, whether evinced in variations of salinity in the oceans, sediment transport by currents, or the formation of atmospheric waves. While it is usually neglected in the study of water wave phenomena, it is interesting to see what sort of effects non-constant density ρ has on water waves. The majority of work on stratified fluids has been devoted to internal waves, generally with fixed boundary, though some notable exceptions — such as the recent papers [29, 30] do exist. Since we are dealing with water, it will be useful to consider the incompressible case. Compressibility adds considerable complexity to questions of density variation, and can usually be neglected for water [24]. In the incompressible case, variations in density will generally have inertial and gravitational effects, and we will see how these effects relate to our analytical results.

In water of constant density there are two explicit solutions for free surface gravity waves with a non-flat surface wave profile: the Gerstner waves and the edge waves found in [3]. The former, first discovered by Gerstner in 1802 (see [15, 16]) and then re-discovered by Rankine in 1863 (see [27]), are two-dimensional rotational waves in infinitely deep water, and represent the only known explicit traveling wave solution to the governing equations for gravity water waves with a free non-flat boundary. Edge waves were first described by Stokes in 1846 (see [28]). They are a type of wave which propagates in the longshore direction, and whose amplitude decays towards the sea. Despite the fact that they are not

as readily observed as customary waves propagating shoreward, edge waves are remarkably common [26]. They play a role in tsunami dynamics on the continental shelf [1, 17, 25] and may partly explain why large tsunamis may induce exceptional wave activity for days afterwards. Additionally, edge waves play a prominent role in coastal sediment transport [14, 20, 23]. In [3] it was shown that Gerstner's flow can be used to construct explicit edge wave solutions propagating along a plane beach, with a two-dimensional free surface that presents variations in the longshore as well as in the offshore direction. In this context we would like to emphasize that in both explicit solutions the water flow has a specific non-zero vorticity and all particle paths are closed. Notice that for irrotational traveling gravity waves there is at most one closed particle path, and this occurs only if there is a strong underlying current (which would have to be uniform in the framework of irrotational flows) cf. [4, 9, 18].

Our aim is to show that it is possible to adapt both solutions to accommodate water of non-constant density presenting an arbitrary stratification.

2. Governing Equations

The governing equations we will consider are the Euler equations with a free boundary, with the addition of a continuity equation that expresses the fact that the density may vary. A mass conservation argument and application of the divergence theorem usually leads to the equation [21]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

referred to either as the equation of mass conservation or equation of continuity. Expanding this, we arrive at

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{u}) + (\mathbf{u} \cdot \nabla)\rho = 0,$$

which reduces to

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0$$

by use of the material derivative. Since we assume our flow to be volume preserving

$$\nabla \cdot \mathbf{u} = 0,$$

and ρ is always non-vanishing, we find that the density of a particle is constant, i.e.

$$\frac{D\rho}{Dt} = 0.$$

We describe Newton's second law (sometimes called the momentum equations) as follows:

$$u_t + uu_x + vv_y + ww_z = -\frac{P_x}{\rho}, \tag{2.1}$$

$$v_t + uv_x + vv_y + wv_z = -\frac{P_y}{\rho}, \tag{2.2}$$

$$w_t + uw_x + vw_y + ww_z = -\frac{P_z}{\rho} - g, \tag{2.3}$$

where $P(x, y, z, t)$ is the pressure and g the gravitational acceleration. In order to decouple the motion of the water and that of the air, we write the dynamic boundary condition

$$P = P_{atm} \quad \text{on the free surface } z = h(x, y, t).$$

The kinematic boundary conditions express the fact that the surface as well as the bed are to be interfaces, that is, there is no motion of water particles normal to these surfaces. The condition on the surface $z = h(x, y, t)$ we write as

$$\frac{D}{Dt}(z - h(x, y, t)) = 0,$$

and for the bed $z = b(x, y, t)$ we write

$$\frac{D}{Dt}(z - b(x, y, t)) = 0.$$

Often one finds a specified bed, such as $z = 0$, or water of infinite depth, where the bottom kinematic condition is replaced by the requirement that the velocity decays as $z \rightarrow -\infty$.

The governing equations as presented above determine the water wave problem under the influence of gravity. We have presented them in Eulerian coordinates, but it will be useful also to describe the problem in Lagrangian coordinates, where we follow a water particle through the flow rather than following the flow from a fixed position. We might imagine we are standing on a bridge above a river: we could cast a wood chip into the stream below and watch its course — this is the Lagrangian perspective. On the other hand, we could fix a sensor to a pole and dip it into the stream, watching how things change at one point in the fluid — this is the Eulerian perspective.

Our Lagrangian independent variables being initial position (a, b, c) and time t , for the position of a particle at time t we write, in general:

$$x(t) = f(a, b, c, t)$$

$$y(t) = h(a, b, c, t)$$

$$z(t) = g(a, b, c, t).$$

Rewriting the momentum equations $\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla(P - gz)$ in terms of Lagrangian variables, we find (with subscripts denoting the partial derivative)

$$\frac{1}{\rho} \frac{\partial}{\partial a}(P + gh) = -f_{tt}f_a - h_{tt}h_a - g_{tt}g_a, \tag{2.4}$$

$$\frac{1}{\rho} \frac{\partial}{\partial b}(P + gh) = -f_{tt}f_b - h_{tt}h_b - g_{tt}g_b, \tag{2.5}$$

$$\frac{1}{\rho} \frac{\partial}{\partial c}(P + gh) = -f_{tt}f_c - h_{tt}h_c - g_{tt}g_c. \tag{2.6}$$

The equation of mass conservation is then expressed by

$$\partial_t \det \begin{pmatrix} f_a & h_a & g_a \\ f_b & h_b & g_b \\ f_c & h_c & g_c \end{pmatrix} = 0.$$

3. Gerstner Waves

Gerstner waves are remarkable in a number of ways — foremost among these is the fact that they are the only known explicit traveling wave solution to the governing equations for gravity water waves. They were discovered in 1802 by Franz-Josef Gerstner (1756–1832), then a professor of mathematics, hydraulics and astronomy in Prague [15]. Gerstner waves are two-dimensional rotational waves in water of infinite depth (see [8] for a discussion of aspects of vorticity in the context of gravity water waves). The water particles describe circular trajectories, whose radius decreases with depth. The form of the free surface is generally a trochoid, or in the extreme case, a cycloid. Although the Gerstner wave has been known for more than 200 years, it was only recently that a rigorous investigation of the entire homogeneous fluid body was performed in [2]. A simplification of this work by means of degree theory was performed in [19] (see also the discussion in [10]). While certain features that are valid for Gerstner waves were subsequently extended to general traveling water waves in a rotational flow (e.g. the fact that profiles that are monotone between crests and troughs have to be symmetric [5, 7, 13] and that the streamlines beneath the surface are real analytic [6]), the fact that all particles move in closed orbits is generally not replicated [12], and the fact that the pressure is constant along streamlines is practically a characterization of Gerstner waves under the assumption that the water has constant density [22].

It is interesting to note the effects that density variation has on different aspects of fluid flow. We will only show that stratification creates vorticity, but further discussion of the inertial and gravitational effects of stratification may be found in [32]. By taking the material derivative of the vorticity ω (or, in another manner of speaking, cross differentiating the momentum equations (2.1) and (2.2)) we find that

$$\frac{D\omega}{Dt} = \frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial x} \frac{\partial P}{\partial y} - \frac{\partial \rho}{\partial y} \frac{\partial P}{\partial x} \right), \quad (3.1)$$

whereby we see that, in general, the vorticity of a particle will change for non-constant density. In fact, we may interpret the term on the right-hand side as $\nabla \rho \times \nabla P$ in the two dimensional setting where we have set the z -components equal to zero. Thus we see that the components of $\nabla \rho$ normal to ∇P create vorticity [34].

This motivates the conclusion of Dubreil–Jacotin [11] that there exist no irrotational traveling waves in a heterogeneous fluid. Furthermore, we see that for barotropic flows (that is, flows where the lines of constant pressure are identical with lines of constant density), the right-hand side above disappears, and the vorticity of a particle is constant.

In fact, as we shall see below, one of the decisive properties of Gerstner waves is that they are indeed barotropic. This first allowed Dubreil–Jacotin [11] to show that these waves are possible solutions for heterogeneous liquids of infinite depth.

Gerstner’s solution is a two-dimensional wave, which allows for a stream function describing the flow — something very useful that cannot generally be found in three dimensions.

As usual, we have the momentum equations

$$u_t + uu_x + vv_y = -\frac{P_x}{\rho}, \tag{3.2}$$

$$v_t + uv_x + vv_y = -\frac{P_y}{\rho} - g, \tag{3.3}$$

where $P(x, y, t)$ is the pressure, and g gravitational acceleration. Note that the role of density in expressing $F = ma$ is better seen when the above equations are written as $\rho \frac{D\mathbf{u}}{Dt} = -\nabla P - \rho g$, the left-hand side expressing mass times acceleration for a unit volume.

To decouple the motion of the water from that of the air, the dynamic boundary condition is

$$P = P_{atm} \quad \text{on the free surface } y = \eta(x, t). \tag{3.4}$$

The surface kinematic condition expresses the fact that the free surface is an interface, i.e. no particles can leave the free surface, and in particular, it is always composed of the same particles. Thus

$$v = \eta_t + u\eta_x \quad \text{on the free surface } y = \eta(x, t).$$

The fact that the water depth is infinite does away with the usual bottom kinematic condition $v = 0$ on $y = -d$. Instead we have the boundary condition

$$\mathbf{u} \rightarrow 0 \quad \text{as } y \rightarrow -\infty.$$

This is to reflect the physical reality that wave motion over deep water does not penetrate very far down [24].

Gerstner’s wave is best described in Lagrangian coordinates. In the two-dimensional Lagrangian perspective we have two markers for a particle, $a \in \mathbb{R}$ and $b \leq b_0 \leq 0$, where b_0 is fixed and represents the still water surface. The label b fixes a particle, while a determines its present location during the motion. Thus, in a sense, we can say that choosing b is like fixing a path-line in the flow.

We write $x = f(a + ct, b)$, $y = h(a + ct, b)$ as above. Then $\frac{D\mathbf{u}}{Dt} = (\ddot{x}, \ddot{y}) = (c^2 f_{aa}, c^2 h_{aa})$. Again changing variables to a, b in (3.2) and (3.3) we find

$$-\frac{1}{\rho} \frac{\partial}{\partial a} (P + gh) = c^2 f_{aa} f_a + c^2 h_{aa} h_a, \tag{3.5}$$

$$-\frac{1}{\rho} \frac{\partial}{\partial b} (P + gh) = c^2 f_{aa} f_b + c^2 h_{aa} h_b. \tag{3.6}$$

In the homogeneous case, we can integrate Eq. (3.5) to give

$$-\frac{P}{\rho} - gh = c^2 f_a^2 + c^2 h_a^2 + C(b),$$

where $C(b)$ is some constant dependent only on b . Differentiating this expression with respect to b and using (3.6) we find

$$c^2 (f_{aa} f_b + h_{aa} h_b - f_a f_{ab} - h_a h_{ab}) = C'(b).$$

The condition of mass conservation can be conveniently rewritten as $\partial_t (f_a h_b - f_b h_a) = 0$.

Since the water is of infinite depth, we will consider the fluid domain $R = \{x \in \mathbb{R}, y \leq b_0 \leq 0\}$, though by periodicity it would suffice to consider one wavelength. We then require $P(a, b_0) = P_{atm}$ as the surface dynamic condition, as well as $(f_a, h_a) \rightarrow 0$ as $b \rightarrow -\infty$. We also have to take into account the surface kinematic condition that there is no velocity component normal to the interface.

Gerstner’s wave can then be represented by

$$x = a + \frac{1}{k} e^{kb} \sin k \left(a + \sqrt{\frac{g}{k}} t \right), \tag{3.7}$$

$$y = b - \frac{1}{k} e^{kb} \cos k \left(a + \sqrt{\frac{g}{k}} t \right). \tag{3.8}$$

It is useful to find expressions for the velocity components u and v . Since $x = x(a + ct, b)$, then $\dot{x} = cx_a$, and likewise for y , we have

$$u = c \left(1 + e^{kb} \cos k \left(a + \sqrt{\frac{g}{k}} t \right) \right), \tag{3.9}$$

$$v = ce^{kb} \sin k \left(a + \sqrt{\frac{g}{k}} t \right), \tag{3.10}$$

where $k = \frac{g}{c^2}$. In particular, since we are in the two-dimensional case, we have the practical advantage of a stream-function formulation, i.e. $\psi_y = u$ and $\psi_x = -v$.

When we plug Gerstner’s solution into the Euler equations in order to get a term for the pressure P , we find a term that depends on the parameter b and the density ρ .

$$P = C - \rho gb + \frac{\rho g}{2k} e^{2kb},$$

where C is some constant (which we will set the atmospheric pressure P_{atm}). To extend the Gerstner wave solution to a liquid of arbitrary stratification, we would like for this pressure term to be entirely independent of the value of a — recall that the free surface is described by (a, b_0) , so this is necessary for the pressure to be constant along the free surface. The special nature of Gerstner waves ensures that this is the case.

Now we know by virtue of incompressibility that the density of a particle must remain constant, i.e. $\frac{D\rho}{Dt} = 0$. Since Gerstner’s wave is a traveling wave, we can shift to a moving coordinate system in which the flow is steady — this means that streamlines and path lines coincide in the moving frame. Thus density is constant along streamlines. Now we need only show that the streamlines depend only on b . In fact, using the stream function above and changing variables to a and b , we find that ψ is independent of a , and in fact that $\psi = c(b - \frac{e^{2kb}}{2k})$. Thus we may say that the density, constant along streamlines, depends only on the parameter b , and therefore the pressure derived from the Euler equations depends likewise entirely on b .

Dubreil–Jacotin [11] demonstrates that the potential and kinetic energy $E = gy - \frac{1}{2} \mathbf{u} \cdot \mathbf{u}$ for Gerstner waves is

$$E = g \left(b - \frac{e^{2kb}}{2k} - \frac{1}{2k} \right).$$

By Bernoulli’s theorem, however, $E = \frac{P}{\rho} + C$ on streamlines, so that the above expresses the pressure and coincides with previous results, e.g. in [2, 35]. The success of Gerstner waves in heterogeneous liquids can be attributed to the fact that the lines of constant density coincide with lines of constant pressure, the streamlines.

Due to the fact that the equations for homogeneous and heterogeneous fluids are otherwise identical, the remaining analysis of Gerstner waves in homogeneous water of [2] or [19] can be transferred to the stratified case. The equation of continuity is satisfied, the boundary conditions hold, and since the proof of the motion of the whole fluid body relies only on properties of the Gerstner map itself — which does not involve the density — we see that this rigorous result also holds in the stratified case.

4. Edge Waves in Stratified Water

There exist a number of models for edge waves, classically based on the linearized equations, with or without the shallow-water assumption [21]. It can be shown that the linear equations of motion have solutions for waves propagating parallel to the beach along a shoreline of constant slope. The wave amplitude dies out farther offshore. It was pointed out by Yih [33] that the Gerstner wave solution to the two-dimensional governing equations for water waves could be extended to the case of three-dimensional edge waves. This was first demonstrated explicitly some 50 years later by Constantin [3].

The idea behind the construction is easily explained. We consider a sloping beach with angle α . Now clearly a sloping beach is not infinitely deep, therefore we introduce new coordinates such that the xy -plane is parallel to the sloping bed, and the z -axis normal to it. In this manner, the fluid domain can be denoted

$$R = \{(x, y, z) \mid x \in \mathbb{R}, y \leq b_0, 0 \leq z \leq (b_0 - y) \tan \alpha\}.$$

The equations of motion (2.1)–(2.3) take on a new form in these coordinates:

$$\frac{Du}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial x} \tag{4.1}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial y} - g \sin \alpha \tag{4.2}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial z} - g \cos \alpha. \tag{4.3}$$

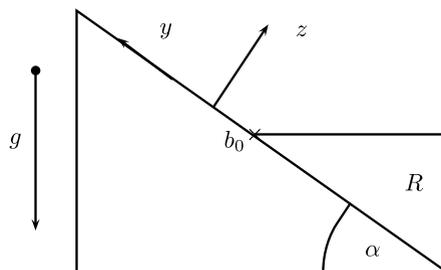


Fig. 1. Coordinate system for edge waves.

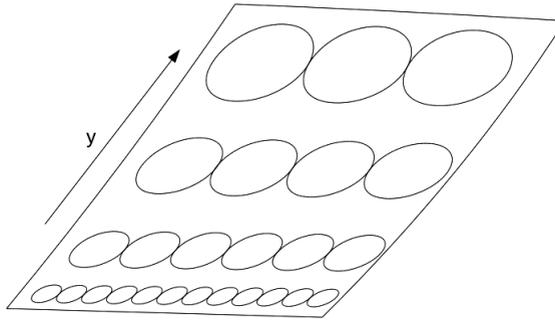


Fig. 2. Circular Gerstner particle paths along a plane in edge wave motion.

The boundary conditions remain the same, where we want no movement normal to the sloping bed. In Lagrangian coordinates, the equations of motion take the form

$$-\frac{1}{\rho} \frac{\partial P}{\partial a} = f_{tt} f_a + (h_{tt} + g \sin \alpha) h_a + (\tilde{g}_{tt} + g \cos \alpha) \tilde{g}_a, \tag{4.4}$$

$$-\frac{1}{\rho} \frac{\partial P}{\partial b} = f_{tt} f_b + (h_{tt} + g \sin \alpha) h_b + (\tilde{g}_{tt} + g \cos \alpha) \tilde{g}_b, \tag{4.5}$$

$$-\frac{1}{\rho} \frac{\partial P}{\partial c} = f_{tt} f_c + (h_{tt} + g \sin \alpha) h_c + (\tilde{g}_{tt} + g \cos \alpha) \tilde{g}_c, \tag{4.6}$$

where we depart from the notation of Sec. 2 by denoting $z(t) = \tilde{g}(a, b, c, t)$ to avoid confusion with the gravitational acceleration g .

The idea is now to take planes of constant z , parallel to the bed, for which $x \in \mathbb{R}, y \leq b_0$, and apply the Gerstner flow construction. This way, on each such slice of the water, we have the particles moving in circular paths, the radius decreasing with y . These Gerstner waves must of course match in the z -direction in order to see a continuous 3-dimensional edge wave. From [3] we find

$$x = a - \frac{1}{k} e^{k(b-c)} \sin(ka + \sqrt{gk \sin \alpha} t), \tag{4.7}$$

$$y = b - c + \frac{1}{k} e^{k(b-c)} \cos(ka + \sqrt{gk \sin \alpha} t), \tag{4.8}$$

$$z = c + c \tan \alpha - \frac{\tan \alpha}{2k} e^{2kb_0} (1 - e^{-2kc(1+\cot \alpha)}). \tag{4.9}$$

It is easy to recognize the Gerstner waves in (4.7) and (4.8) above. We refer to [3] for a detailed discussion of edge waves, noting only that their amplitude decays away from the beach, and that their phase velocity is considerably slower than that of shallow water waves or deep water waves of the same frequency.

An argument similar to that used for Gerstner waves in Sec. 3 yields a system of equations for the pressure P . Notably, we find that the pressure in the z -direction is hydrostatic, i.e.

$$\frac{\partial P}{\partial z} = -\rho g \cos \alpha.$$

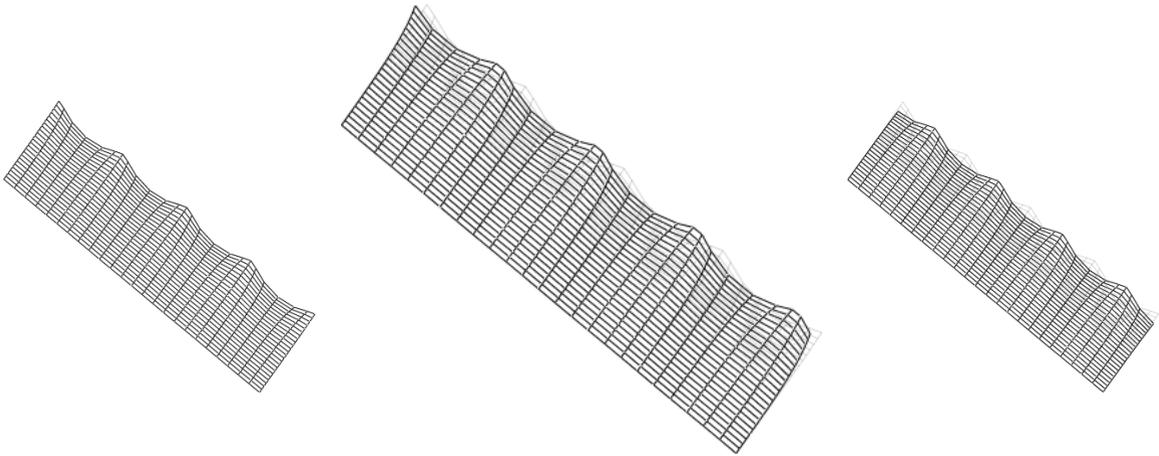


Fig. 3. Edge wave motion in longshore direction. Shown propagating to the left, the beach is on the upper right-hand side of each image.

These equations may be solved to yield

$$P = P_0 + \frac{\rho g \sin \alpha}{2k} e^{2k(b-c)} - \rho g (c \cos \alpha + (b - b_0) \sin \alpha) - \frac{\rho g \sin \alpha}{2k} e^{-2kc(1+\cot \alpha)+2kb_0}.$$

We also see that the pressure is independent of a , and since the density ρ is independent of a on any plane of constant z , we see that P/ρ is independent of a as in the case of Gerstner waves. This means that edge waves are also possible for arbitrary density stratification.

As in the two-dimensional case, we can see that density variation may create vorticity. Taking the material derivative of the vorticity, we see that $\frac{D\omega}{Dt} = \frac{1}{\rho^2} (\nabla \rho \times \nabla P)$, and again we see that edge waves, like Gerstner waves, are barotropic.

5. Concluding Remarks

Recent results have established the rigorous evolution of the fluid domain under Gerstner’s wave [2, 19], and provided explicit descriptions of edge waves [3]. We have seen that this recent work can be simply extended to a liquid of arbitrary stratification. The arguments involved in this extension are primarily centered around finding a suitable pressure, and showing that the structure of the particle trajectories — based on the Gerstner map — is such that lines of constant pressure are identical with lines of constant density. Since we consider gravitational waves, the only physically relevant cases of stratification are those which are gravitationally stable, meaning that the density is non-decreasing with depth (see [34] for a discussion of some results on instability in stratified fluids). Our results update previous work in this vein by Dubreil–Jacotin [11] and Yih [31].

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