Ordinary Differential Equations - 10413181

Homework No. 5

1. **Parameter–dependent ODEs and bifurcation**

   We have discussed (and practiced) drawing direction fields for autonomous equations

   $$y' = f(y).$$

   Recall that the values of y where $f(y) = 0$ are "flat" trajectories in the direction field (so-called fixed–points or equilibrium points). Given the autonomous ODE

   $$\frac{dx}{dt} = r + x^2 \text{ where } r \in \mathbb{R} \text{ is a parameter} \tag{1}$$

   (a) Solve the ODE for $r > 0$, $r < 0$, and $r = 0$. *(You may choose $r = 1, -1,$ and 0 to simplify things, or leave r as is. Feel free to disregard constants of integration, since this is an autonomous equation.)*

   (b) (Optional) *Think about* the qualitative behaviour of each of the solutions. You may sketch a graph of the solution to better visualize things.

   (c) Call the right-hand side of the equation $f(x) = r + x^2$.

   - Plot the graph $(x, f(x))$ for $r = 1, -1$ and 0. (*Don't overthink it. This is just like in high-school!)*
   - At how many points does the graph intersect the x-axis in each case? (*Equivalently, how many zeroes does $f(x)$ have for the different values of r?*
   - Sketch (with the help of the above answer) a direction field in the (x, t)–plane for equation (1) and $r = 1, -1$ and 0. Where are the fixed points?
   - Compare the dynamics you see in your sketches with the analytical solutions. Also: compare the effort needed to get a qualitative picture from the direction field to that needed to write down analytical solutions.

- (Optional) Note that as the parameter r changes, fixed points appear or disappear. This phenomenon is called *bifurcation*!
- (Really Optional!) If you want to find out more about this, try the same exercise as above on the equation $x' = rx - x^2$. How does the stability of the fixed points change (looking at the direction field)? (This is the *trans-critical bifurcation* in contrast to the *saddle-node bifurcation* above).

2. **Practice with second order, homogeneous, constant coefficient ODES**

   For the following problems: (a) Substitute $y = e^{rt}$ into the equations. (b) Find the root(s) of the resulting second order polynomial. (c) Convince yourself that $Ce^{r_1 t}$ for r_1 any root solves the equation. (d) Write down the general solution $C_1 e^{r_1 t} + C_2 e^{r_2 t}$ like in class.

   (a) $y'' + 3y' - 4y = 0$

   (b) $y'' + 5y' = 0$

   (c) $y'' + 3y' + 2y = 0$

   For the next equations, also determine the value of the two constants that satisfy the initial value problems:

   (d) $y'' - 3y' + 2y = 0, \quad y(0) = 1,\ y'(0) = 1$

   (e) $y'' + 3y' = 0, \quad y(0) = -2,\ y'(0) = 3$