Ordinary Differential Equations - 10413181

Homework No. 10 – Solutions

1. Given the homogeneous ODE $t^2y'' - 2y = 0$, which is of Cauchy-Euler type, we can make a substitution $y = t^r$ to find solutions of this specific form. Hence the ODE is transformed into $t^2r(r-1)t^{r-2} - 2t^r = 0 \Leftrightarrow t^r(r(r-1)-2) = 0$ which is equivalent to $r^2 - r - 2 = 0$ and has roots $r = 2, r = 1$. It is then easy to check that $y_1 = t^2$ and $y_2 = 1/t$ are a fundamental set of solutions with Wronskian $W = -3$.

   We have not shown any use of the method of undetermined coefficients for non-constant coefficients (you may try and see that you run into problems). But, since we already have a fundamental set of solutions, it is attractive to try the method of variation of parameters. **Remember:** this method works on the **normalized** equation with leading coefficient 1!

   $$y'' - \frac{2}{t^2} = 1$$

   Substituting gives us the matrix equation

   $$\begin{pmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

   Hence

   $$u_1' = \frac{1}{3}(t^{-1}) \Rightarrow u_1 = \frac{1}{3}\ln(t)$$
   $$u_2' = -\frac{1}{3}(t^2) \Rightarrow u_2 = -t^3$$

   Thus $y_1u_1 + y_2u_2 = \frac{t^2}{3}\ln(t) - t^2$ leads to the general solution

   $$y = At^2 + \frac{B}{t} + \frac{t^2}{3}\ln(t).$$

   Another possible method of solution is reduction of order, say using $y = t^2v(t)$ and deriving the first order equation $4tu + t^2u' = 1$ for $u = v'$.

2. Recall that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely if (for $a_n \neq 0$)

   $$\lim\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0| \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-x_0|\cdot L \text{ and } |x-x_0| < 1/L.$$

(a)

$$|x-2|\lim|\frac{1}{1}| = |x-2|\cdot 1$$

Hence the radius of convergence is 1.

(b)

$$\lim\frac{n+1}{n}\frac{2^n}{2^{n+1}} = \frac{1}{2}$$

Hence the radius of convergence is 2.

(c)

$$\lim|\frac{2^{n+1}(x+1/2)^{n+1}n^2}{(n+1)^2 2^n (x+1/2)^n}| = (2x+1)\lim|\frac{n^2}{(n+1)^2}| = (2x+1)$$

Hence the radius of convergence is 1/2.

3. Given $y'' - xy' - y = 0$ and expanding about $x_0 = 0$:

$$y = \sum_{n=0}^{\infty} a_n x^n$$
$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$
$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

Hence the equation becomes

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

This leads to

$$2a_2 - a_0 = 0$$
$$(n+2)(n+1)a_{n+2} - (n+1)a_n = 0$$

and so the recursion relation

$$a_{n+2} = \frac{a_n}{n+2}.$$

We have the fundamental set of solutions

$$y_1(x) = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \ldots = \sum_{n=0}^{\infty}\frac{x^{2n}}{2^n n!}$$
$$y_2(x) = x + \frac{x^3}{3} + \frac{x^5}{15} + \ldots = \sum_{n=0}^{\infty}\frac{2^n n! x^{2n+1}}{(2n+1)!}$$

Substituting $\psi = 1 + x + \frac{x^2}{2} + \frac{x^3}{3}$ into the equation yields:

$$1 + 2x - x - x^2 - x^3 - 1 - x - \frac{x^2}{2} - \frac{x^3}{3} = -\frac{3x^2}{2} - \frac{4x^3}{3},$$

so that only terms of order x^2 remain (these are small if we are sufficiently close to the expansion point $x_0 = 0$).