Ordinary Differential Equations - 10413181

Homework No. 11 – Solutions

1. $y'' + \cos(x)y = 0$ is given. We make the power series ansatz (about $x_0 = 0$)

$$y = \sum_{n=0}^{\infty} a_n x^n, \text{ so that } y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

Using the Taylor series of cos about zero,

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots$$

we see

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = -\sum_{n=0}^{\infty} a_n x^n \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots\right).$$

Collecting terms, we find

$$\begin{aligned} 2a_2 &= -a_0 \\ 6a_3 &= -a_1 \\ 12a_4 &= a_0 \end{aligned}$$

which suffices to write down the first four terms of the solution:

$$y = a_0 + a_1 x - \frac{a_0}{2}x^2 - \frac{a_1}{6}x^3 + \frac{a_0}{12}x^4 + \dots$$

In principle this could be continued as long as desired.

2. Note

$$\sum_{n=0}^{\infty} x^n := \lim_{N\to\infty} \sum_{n=0}^{N} x^n.$$

Hence

$$\begin{aligned} &(1-x)\sum_{n=0}^{\infty} x^n = (1-x)\lim_{N\to\infty}\sum_{n=0}^{N} x^n = \lim_{N\to\infty}(1-x)\sum_{n=0}^{N} x^n \\ &= \lim_{N\to\infty}(1-x)(1+x+x^2+\dots+x^N) \\ &= \lim_{N\to\infty}(1+x\dots+x^N - x - x^2 - \dots - x^{N+1}) \\ &= \lim_{N\to\infty}(1-x^{N+1}) = 1. \end{aligned}$$

3. (a)

$$u'' + 2u' + 2u = 0$$
$$v := u' \Rightarrow v' = u'' = -2u' - 2u = -2v - 2u$$
$$\begin{cases} u' = v \\ v' = -2v - 2u \end{cases}$$

(b)

$$t^2u'' + tu' + (t^2 - 1)u = 0$$
$$v := u' \Rightarrow v' = u'' = -\frac{u'}{t} - \frac{(t^2-1)u}{t^2} = -\frac{v}{t} - \frac{(t^2-1)u}{t^2}$$
$$\begin{cases} u' = v \\ v' = -\frac{v}{t} - \frac{(t^2-1)u}{t^2} \end{cases}$$

(c)

$$u'''' - u = 0$$
$$u' =: v, \quad v' =: w = u'', \quad w' =: z = u''', \quad z' = u'''' = u$$
$$\begin{cases} u' = v \\ v' = w \\ w' = z \\ z' = u \end{cases}$$

4. The system is equivalent to the following in matrix form:

$$\begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \\ 4 \end{pmatrix}.$$

Inverting the matrix leads to

$$x_1 = 2\frac{1}{3}, \quad x_2 = -1\frac{1}{3}, \quad x_3 = \frac{2}{3}.$$

5. (a) The characteristic polynomial associated with matrix A is $\lambda^2 - \lambda - 2 = 0$ which has roots 2 and -1.

   (b) The characteristic polynomial associated with matrix B is $\lambda^2 - 2\lambda - 35 = 0$ which has roots 7 and -5.