Ordinary Differential Equations - 10413181

Homework No. 12 – Solutions

1.

$$W(x^1, x^2) = \det \begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix} = t^2.$$

These are linearly independent everywhere except $t = 0$.

2.

$$W(x^1, x^2) = \det \begin{pmatrix} t & e^t \\ 1 & e^t \end{pmatrix} = e^t(1 - t).$$

These are linearly independent everywhere except $t = 1$.

3. There are two steps to checking that this is a fundamental set of solutions. The first is to check that each of these are in fact **solutions!** Denoting the matrix by A:

$$\frac{d}{dt}x^1 = e^{2t} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = Ax^1$$

$$\frac{d}{dt}x^2 = e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = Ax^2$$

$$\frac{d}{dt}x^3 = e^{-t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = Ax^3$$

Subsequently we need to check that they are a fundamental set, for which we employ the Wronskian:

$$W(x^1, x^2, x^3) = det \begin{pmatrix} e^{2t} & e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & -e^{-t} & -e^{-t} \end{pmatrix} = 3 \neq 0$$

4. For the matrix

$$A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$$

we have associated a characteristic polynomial $\lambda^2 - 6\lambda + 8 = 0$. This has roots $\lambda_1 = 4$, $\lambda_2 = 2$. Eigenvectors $\mathbf{x}$ to eigenvalue λ_1 must fulfill $(\mathbf{A} - 2\mathbf{1})\mathbf{x} = 0$ or

$$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $3x_1 = x_2$ and an eigenvector is given by

$$\xi_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Eigenvectors $\mathbf{x}$ to eigenvalue λ_2 must fulfill $(\mathbf{A} - 4\mathbf{1})\mathbf{x} = 0$ or

$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $x_1 = x_2$ and another eigenvector is given by

$$\xi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The general solution is then written in the form

$$\mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for arbitrary constants c_1, c_2 to be determined by the initial conditions. Given the initial condition

$$\mathbf{x}(t = 0) = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

we find

$$c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

or equivalently

$$\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

Inverting the matrix then yields $c_1 = -3$ and $c_2 = 7$. The solution to the IVP is thus

$$-3e^{2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 7e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$