Homework 2 Solutions

1. (a) $y' - y = y^2$ is a separable equation, but it can also be identified as a Bernoulli equation with $n = 2$. This invites the substitution $y^-1 = v \Leftrightarrow y = 1/v$. Using the Chain Rule, we find $y' = -v^{-2}v'$, so that the equation is transformed into

$$-v' - v = 1,$$

a linear equation! Now either the integrating factor or separation methods apply. For example, using separation yields

$$v = Ce^{-x} - 1.$$

Finally the solution must be written in the original variables, via the transform $y = 1/v$, so that

$$y = \frac{1}{Ce^{-x} - 1}$$

(b) This nonlinear, nonseparable equation may be identified as a Bernoulli equation with exponent $n = -2$. This invites the transformation $v = y^3$, or upon applying the chain rule, $y' = (1/3)v^{-2/3}v'$. This transforms the equation into the linear equation

$$v' - \frac{6}{x}v = 3x$$

Now we can use the integrating factor method with $\mu = x^{-6}$ to give

$$v = -\frac{3}{4}x^2 + Cx^6,$$

and finally

$$y = (Cx^6 - \frac{3}{4}x^2)^{1/3}.$$

(c) $t^2y' + 2ty - y^3 = 0$ is a Bernoulli equation with $n = 3$. The substitution is thus $y^{-2} = v$ and $y' = -(1/2)v^{-3/2}v'$. Plugging this into the equation reduces it to the linear equation

$$v' - \frac{4}{t}v + \frac{2}{t^2} = 0.$$

This can be solved via the integrating factor method, where the integrating factor $\mu = t^{-4}$. Hence

$$v = \frac{2}{5t} + Ct^4 \Rightarrow y = \frac{1}{(2/(5t) + Ct^4)^{1/2}}$$

2. (a) $y' = 1 + t^2 - 2ty + y^2$ has the particular solution $y_1(t) = t$. The Ricatti ansatz $y = t + 1/v$ means $y' = 1 - v^{-2}v'$, which is substituted into the equation to reduce it to

$$v' = -1$$

a great simplification. Thus $v = C - t$, and

$$y(t) = t + \frac{1}{C - t}.$$

(b) This Ricatti equation has $y_1(t) = 1/t$, so that

$$y' = \frac{-1}{t^2} - \frac{1}{v^2}v'$$

which is plugged in to the equation to give the linear equation

$$v' + \frac{v}{t} = -1.$$

This has an integrating factor of $\mu = t$, so that $v = -t/2 + C/t$. Finally

$$y = \frac{1}{t} + \frac{2t}{C - t^2}$$

after a bit of algebra.

3. (a) Since the equation is homogeneous, $y = 0$ is a solution to the IVP $y(0) = 0$.

(b) The ODE is nonlinear, so we may separate it to give

$$\frac{1}{y^{1/3}}y' = \sin(2t).$$

Subsequently, computing the integrals

$$\frac{3y^{2/3}}{2} = \frac{1 - \cos(2t)}{2} = \sin^2(t),$$

so that the two other solutions to this initial value problem are

$$y = \pm\sqrt{8/27}\,\sin^3(t).$$

(c) The non-uniqueness may be traced back to the failure of

$$\frac{\partial}{\partial y}(y^{1/3}\sin(2t))$$

to be continuous at zero, so that the hypotheses of the Existence & Uniqueness Theorem are not satisfied.