Homework 5 Solutions

1. Parameter–dependent ODEs and bifurcation

Given the first order, nonlinear, separable ODE

$$\frac{dx}{dt} = r + x^2$$

with a parameter $r \in \mathbb{R}$, we have several options of investigating the behavior of solutions. We shall see that the most straightforward (namely solving the equation) is the hardest and gives the least insight. This is a standard situation, and one reason why qualitative analysis of ODEs (especially those for which an explicit solution cannot be found) is so valuable.

Step 1: Solving the ODEs

$$x' = r + x^2 \Rightarrow \int \frac{dx}{r + x^2} = t + C$$

1. ($r = -a^2 < 0$)

$$\int \frac{dx}{x^2 - a^2} = \begin{cases} \frac{1}{2a} \ln(\frac{x-a}{x+a}) & x^2 > a^2 \\ \frac{1}{2a} \ln(\frac{a+x}{a-x}) & x^2 < a^2 \end{cases} \tag{1}$$

Hence either

$$\frac{x-a}{x+a} = Ce^{2at} \Rightarrow x = a\frac{1 + Ce^{2at}}{1 - Ce^{2at}}$$

or

$$\frac{a+x}{a-x} = Ce^{2at} \Rightarrow x = a\frac{Ce^{2at} - 1}{1 + Ce^{2at}}$$

2. ($r = 0$)

$$\int \frac{dx}{x^2} = -x^{-1} \tag{2}$$

Hence

$$x = \frac{1}{C - t}$$

3. ($r = a^2 > 0$)

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan(x/a) \tag{3}$$

Hence

$$x = a \tan(at + C)$$

It is rather hard to get information about the dependence of each of these solutions on the parameter r. Instead, we may take a different approach: first we shall plot the right-hand side of the equation, $f(y) = r + x^2$ for the three cases:

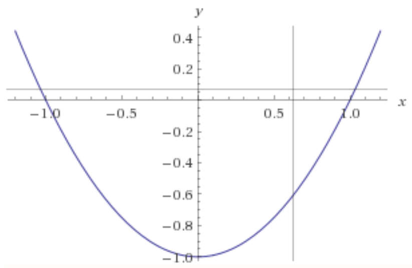

Figure 1: $r < 0$

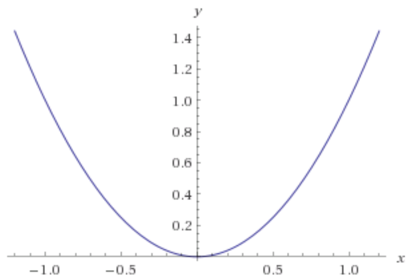

Figure 2: $r = 0$

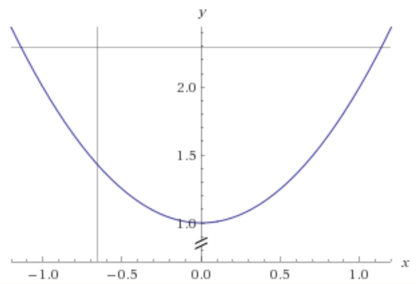

Figure 3: $r > 0$

Note that each zero of $f(y)$ is a *equilibrium point* where $x' = 0$! We can easily see that decresing the parameter from $r = 1$ (where there are no equilibrium solutions) creates a single equilibrium point when $r = 0$, which further splits off into two equilibrium points for $r < 0$. These basic dynamics are captured from the direction fields:

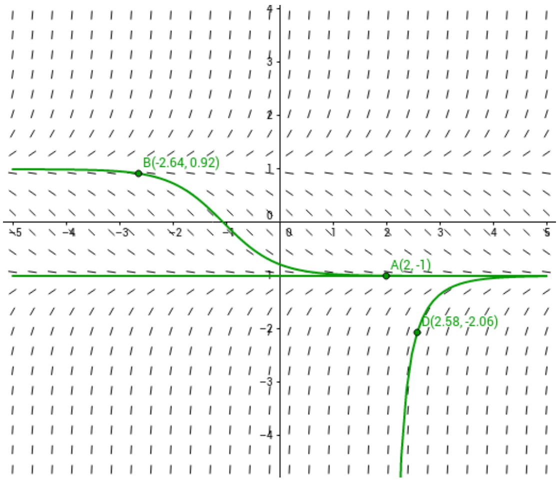

Figure 4: $r < 0$

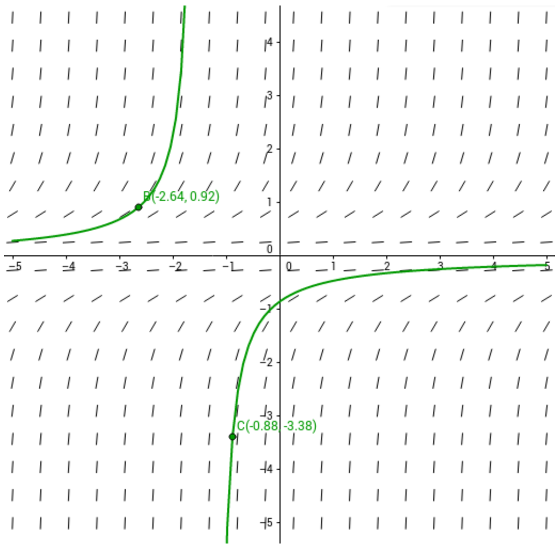

Figure 5: $r = 0$

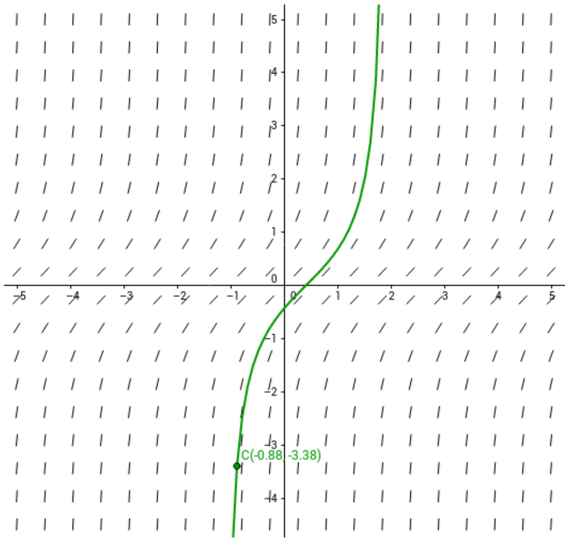

Figure 6: $r > 0$

This phenomenon of fixed points appearing or disappearing (or stability shifting) is known as bifurcation. Note that the fixed point appearing at $r = 0$ is semi-stable, and of the two fixed points appearing after bifurcation one is stable and the other unstable.

For more information, look at the excellent book *Nonlinear Dynamics and Chaos* by Steven Strogatz.

Problem 2. (a)

$$\begin{aligned} r^2 + 3r - 4 = 0 &\Rightarrow r_1 = 1, r_2 = -4 \\ &\Rightarrow y = C_1 e^t + C_2 e^{-4t} \end{aligned}$$

(b)

$$\begin{aligned} y'' + 5y' = 0 \Rightarrow r^2 + 5r = 0 &\Rightarrow r_1 = 0, r_2 = -5 \\ &\Rightarrow y = C_1 e^{-5t} + C_2 \end{aligned}$$

(c)

$$\begin{aligned} r^2 + 3r + 2 = 0 &\Rightarrow r_1 = -1, r_2 = -2 \\ y &= C_1 e^{-t} + C_2 e^{-2t} \end{aligned}$$

(d)

$$\begin{aligned} r^2 - 3r + 2 = 0 \Rightarrow r_1 = 1, r_2 = 2 \\ \Rightarrow y = C_1 e^t + C_2 e^{2t} \\ y(0) = C_1 + C_2 = 1 \\ y'(0) = C_1 + 2C_2 = 1 \\ \Leftrightarrow C_1 = 1, C_2 = 0 \Rightarrow y = e^t \end{aligned}$$

(e)

$$\begin{aligned} r^2 + 3r = 0 \Rightarrow r_1 = -3, r_2 = 0 \\ \Rightarrow y = C_1 e^{-3t} + C_2 \\ y(0) = C_1 + C_2 = -2 \\ y'(0) = -3C_1 = 3 \Rightarrow C_1 = -1 \Rightarrow C_2 = -1 \\ \Rightarrow y = -e^{-3t} - 1 \end{aligned}$$