Ordinary Differential Equations - 10413181

Homework No. 6

1. (a) $y'' + 4y = 0, \quad y_1(t) = \cos(2t), y_2(t) = \sin(2t).$

   Calculate $y_1'' = -4y_1, \quad y_2'' = -4y_2$ So both are solutions.

$$W = y_1 y_2' - y_2 y_1' = \cos(2t) 2\cos(2t) - \sin(2t)(-2)\sin(2t) = 2 \neq 0$$

   (b) $x^2 y'' - x(x+2)y' + (x+2)y = 0$, $(x > 0)$, $\quad y_1(x) = x$, $y_2(x) = xe^x$

   Calculate $y_1' = 1, y_1'' = 0$ so that

$$x^2 y_1'' - x(x+2)y_1' + (x+2)y_1 = 0 - x(x+2) + (x+2)x = 0.$$

   Also: $y_2' = e^x + xe^x, y_2'' = 2e^x + xe^x$ so that

$$x^2 y_2'' - x(x+2)y_2' + (x+2)y_2 = x^2(2e^x + xe^x) - x(x+2)(e^x + xe^x) + (x+2)xe^x = 0$$

   Finally the Wronskian

$$W = x(e^x + xe^x) - xe^x = x^2 e^x \neq 0 \quad (x > 0).$$

2. For the equation

$$y'' - y' - 2y = 0$$

   show that $y_1 = e^{-t}$, $y_2 = e^{2t}$ are a fundamental set of solutions.

   Here we calculate the Wronskian

$$W = e^{-t} 2e^{2t} + e^{-t} e^{2t} = 3e^t \neq 0$$

   (a) Note that

$$y_3 = -2y_2$$

$$y4 = y_1 + 2y_2$$

$$y_5 = 2y_1 + 4y_2$$

   Since the equation is linear, and we have checked y_1 and y_2 are solutions, so all linear combinations are solutions too. Hence we do not need to do anything further!

(b) Here we recast the pairs by noting which is a linear combination of another solution:

$$(y_1, y_3) \Leftrightarrow (y_1, -2y_2) \text{ is a fundamental set}$$

$$(y_2, y_3) \Leftrightarrow (y_2, -2y_2) \text{ is not a fundamental set}$$

$$(y_1, y_4) \Leftrightarrow (y_1, y_1 + 2y_2) \text{ is a fundamental set}$$

$$(y_4, y_5) \Leftrightarrow (y_1 + 2y_2, 2(y_1 + 2y_2)) \text{ is not a fundamental set}$$

Of course one can also check the Wronskians all by hand.

3. Determine the Wronskian of two solutions of the following equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

Rewriting in normal form so that the coefficient of y'' is 1, use Abel's formula to find

$$W = Ce^{-\int 1/x dx} = C/x$$

4. Find the general solution to the following equation:

$$y'' - 2y' + 2y = 0$$

Substitute $y = e^{rt}$ to get a characteristic polynomial

$$r^2 - 2r + 2 = 0$$

with roots

$$r = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$

This gives a solution

$$y = C_1 e^{t(1+i)} + C_2 e^{t(1-i)}$$

or

$$y = C_1 e^t(\cos t + i \sin t) + C_2 e^t(\cos t - i \sin t)$$

or

$$y = e^t(A \cos t + B \sin t).$$