Ordinary Differential Equations - 10413181

Homework No. 7

1. For the following expressions (with $t \in \mathbb{R}$), identify the real part, the imaginary part, the complex conjugate, and use Euler's formula to write each as $a + ib$

   (a) $\exp((1+2i)t)) = e^t e^{2it}$

   Complex conjugate: $e^t e^{-2it}$

   Euler's formula: $e^t(\cos(2t) + i\sin(2t))$

   Real part: $e^t \cos(2t)$

   Imaginary part: $e^t \sin(2t)$

   (b) $\exp((2+4i)t)) = e^{2x} e^{4ix}$

   Complex conjugate: $e^{2x} e^{-4ix}$

   Real part: $(z + \bar{z})/2 = e^{2x}\cos(4x)$

   Imaginary part: $(z - \bar{z})/2i = e^{2x}\sin(4x)$

   Euler's formula: $e^{2x}(\cos(4x) + i\sin(4x))$

   (c) $e^{2+i\pi/2} = e^2 e^{i\pi/2}$

   Complex conjugate: $e^2 e^{-i\pi/2}$

   Euler's formula: $e^2(\cos(\pi/2) + i\sin(\pi/2) = ie^2\sin(\pi/2)$

   Real part: 0

   Imaginary part: $e^2(\sin(\pi/2))$

2. Find solutions to the following initial value problems, write these using Euler's formula, and describe their behavior for large time t

   (a) $y'' + 4y = 0, \quad y(0) = 0, y'(0) = 1$

   Characteristic polynomial $r^2 + 4 = 0$ has roots $r = \pm 2i$

   $y = Ae^{2it} + Be^{-2it}$

   $y(0) = A + B = 0 \Rightarrow A = -B$

   $y'(0) = 2iA - 2iB = 1 \Rightarrow B = i/4$

   $\Rightarrow y = -ie^{2it}/4 + ie^{-2it}/4 \Leftrightarrow y = \sin(2t)/2$

   (b) $y'' + y' + 5y = 0, \quad y(0) = 1, y'(0) = 0$

   Characteristic polynomial $r^2 + 4r + 5 = 0$ has roots $r = -2 \pm i$

$\Rightarrow y = Ae^{(-2+i)t} + Be^{(-2-i)t} = e^{-2t}(C\cos(t) + D\sin(t))$

$y(0) = C = 1$

$y'(0) = -2C + D = 0 \Rightarrow D = 2$

$\Rightarrow y = e^{-2t}(\cos(t) + 2\sin(t))$

(c) $y'' - 6y' + 13y = 0, \quad y(\pi/2) = 0, y'(\pi/2) = 2$

Characteristic polynomial $r2 - 6r + 13 = 0$ has roots $3 \pm 2i$

$\Rightarrow y = e^{3t}(C_1 \sin(2t) + C_2 \cos(2t))$

$y(\pi/2) = e^{3\pi/2}(-C_2) = 0 \Rightarrow C_2 = 0$

$y'(\pi/2) = -e^{3\pi/2} 2C_1 = 2 \Rightarrow C_1 = -e^{3\pi/2}$

$\Rightarrow y = -e^{3\pi/2}\sin(2t)$

3. (Optional) Pick one of the problems above and determine the envelope and (pseudo-) frequency of the oscillation, if appropriate.

4. In class we saw that certain kinds of second-order ODEs with non-constant coefficients can be reduced to the constant coefficient case with a suitable transformation. For the Cauchy-Euler type equation

$$t^2y'' + 4ty' + 2y = 0$$

use the substitution $x = \ln(t)$ to calculate dy/dt, d^2y/dt^2 and solve the equation.

The substitution $x = \ln(t)$ implies

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{t}\frac{dy}{dx}, \quad \frac{d^2y}{dt^2} = \frac{-1}{t^2}\frac{dy}{dx} + \frac{1}{t^2}\frac{d^2y}{dx^2}$$

Substituting into the equation then yields the constant coefficient equation

$$y'' + 3y' + 2y = 0$$

with characteristic polynomial $r^2 + 3r + 2 = 0$ with roots $r = -2, -1$. Hence

$$y = Ae^{-2x} + Be^{-x} = Ae^{-2\ln(t)} + Be^{-\ln(t)} = \frac{A}{t^2} + \frac{B}{t}.$$