Ordinary Differential Equations Practice Questions

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1 Introductory thoughts about ODEs

Determine the value of r for which the following ODEs have solutions of the form $y = e^{rt}$:

1. $y' + 3y = 0$
2. $y'' - y = 0$
3. $y'' + y' - 6y = 0$
4. $y''' - 4y'' + 3y' = 0$

Determine the values of r for which the following ODEs have solutions of the form $y = t^r$ for $t > 0$

1. $t^2y'' + 4ty' + 2y = 0$
2. $t^2y'' - 4ty' + 4y = 0$

2 Linear equations and the integrating factor method

Find the solutions to the following ODEs and initial value problems

1. $y' + 3y = t + e^{-2t}$
2. $y' - 2y = t^2e^{2t}$
3. $y' + y = te^{-t} + 1$
4. $y' + \frac{y}{t} = 3\cos(2t)$, $t > 0$
5. $y' - 3y = 4e^t$
6. $ty' + 2y = \sin(t)$, $t > 0$
7. $y' + 2ty = 2te^{-t^2}$
8. $(1 + t^2)y' + 4ty = (1 + t^2)^{-2}$
9. $ty' - y = t^2e^{-t}$, $t > 0$
10. $2y' + y = 3t$
11. $y' + y = 5\sin(2t)$
12. $2y' + y = 3t^2$
13. $y' - y = 2te^{2t}$, $y(0) = 1$
14. $y' + 3y = te^{-3t}$, $y(1) = 0$
15. $ty' + 2y = t^2 - t + 1$, $y(1) = 1/2$, $t > 0$
16. $y' + \frac{2y}{t} = \frac{\cos(t)}{t^2}$, $y(\pi) = 0$, $t > 0$
17. $y' - 4y = e^{4t}$, $y(0) = 2$
18. $ty' + 2y = \sin(t)$, $y(\pi/2) = 1$, $t > 0$
19. $t^3y' + 4t^2y = e^{-t}$, $y(-1) = 0$, $t < 0$
20. $ty' + (t + 1)y = t$, $y(\ln 2) = 1$, $t > 0$

3 Separable ODEs

Solve the given ODEs and initial value problems

1. $y' = x^2/y$
2. $y' + y^2 \sin(x) = 0$
3. $y' = x^2/y(1+x^3)$
4. $y' = (3x^2-1)/(3+2y)$
5. $y' = (\cos^2(x))(\cos^2(2y))$
6. $xy' = (1-y^2)^{1/2}$
7. $y' = (x-e^{-x})/(y+e^y)$
8. $y' = (1+y^2)^{-1}x^3$
9. $y' = (1-2x)y^2$, $y(0) = -1/6$
10. $y' = (1-2x)/y$, $y(1) = -2$
11. $xdx + ye^{-x}dy = 0$, $y(0) = 1$
12. $y' = xy^3(1+x^2)^{-1/2}$, $y(1) = -2$
13. $y' = 2x/(y+x^2y)$, $y(0) = -2$
14. $dr/d\theta = r^2\theta$, $r(1) = 2$
15. $y' = 2x/(1+2y)$, $y(2) = 0$

4 Exact equations

Determine whether the following ODEs are exact. If so, find the solution.

1. $(2x+4y) + (2x-2y)y' = 0$
2. $(2x+3) + (2y-2)y' = 0$
3. $(3x^2-2xy+2)dx + (6y^2-x^2+3)dy = 0$
4. $(2xy^2+2y) + (2x^2y+2x)y' = 0$
5. $y' = -\frac{ax+by}{bx+cy}$
6. $y' = -\frac{ax-by}{bx-cy}$
7. $(e^x \sin(y) - 2y\sin(x))dx + (e^x\cos(y) + 2\cos(x))dy = 0$
8. $(e^x\sin(y) + 2y)dx - (3x - e^x\sin(y))dy = 0$
9. $(ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x)dx + (xe^{xy}\cos(2x) - 3)dy = 0$
10. $(x\ln(y) + xy)dx + (y\ln(x) + xy)dy = 0$, $x > 0$, $y > 0$
11. $(y/x + 6x)dx + (\ln(x) - 2)dy = 0$, $x > 0$
12. $\frac{xdx}{(x^2+y^2)^{3/2}} + \frac{ydy}{(x^2+y^2)^{3/2}} = 0$

5 Second order equations with constant coefficients: real roots

Solve the given ODEs and initial value problems

1. $y'' + 3y' - 4y = 0$
2. $2y'' - 3y' + y = 0$
3. $6y'' - y' - y = 0$
4. $y'' + 3y' + 2y = 0$
5. $y'' + 5y' = 0$
6. $9y'' - 16y = 0$
7. $y'' - 9y' + 9y = 0$
8. $y'' - 2y' - 2y = 0$
9. $y'' - 3y' + 2y = 0, \quad y(0) = 1, \, y'(0) = 1$
10. $y'' + 4y' + 3y = 0, \quad y(0) = 2, \, y'(0) = -1$
11. $y'' + 5y' + 3y = 0, \quad y(0) = 1, \, y'(0) = 0$
12. $y'' + 3y' = 0, \quad y(0) = -2, \, y'(0) = 3$
13. $6y'' - 5y' + y = 0, \quad y(0) = 4, \, y'(0) = 0$
14. $2y'' + y' - 4y = 0, \quad y(0) = 0, \, y'(0) = 1$
15. $y'' + 8y' - 9y = 0, \quad y(1) = 1, \, y'(1) = 0$
16. $4y'' - y = 0, \quad y(-2) = 2, \, y'(-2) = -1$

6 Second order equations: Wronskian and fundamental solution sets

Find the Wronskian of the following function pairs

1. e^{2t}, e^{-3t}
2. $\cos(t)$, $\sin(t)$
3. e^{-2t}, te^{-2t}
4. x, xe^{x}
5. $e^{t}\sin(t)$, $e^{t}\cos(t)$
6. $\sin^2(\theta)$, $1 - \cos(2\theta)$

Find the longest interval in which the given initial value problems have a unique, twice-differentiable solution (Consider the hypotheses of the E & U Theorem!)

1. $ty'' + 3y = t, \quad y(2) = 1, y'(2) = 3$
2. $(t-1)y'' - 3ty' + 4y = \sin(t), \quad y(-2) = 2, y'(-2) = 1$
3. $t(t-4)y'' + 3ty' + 5y = 2, \quad y(3) = 0, y'(3) = -1$
4. $y'' + \cos(t)y' + 3\ln(|t|)y = 0, \quad y(3) = 2, y'(3) = 1$
5. $(x-3)y'' + xy' + \ln(|x|)y = 0, \quad y(1) = 0, y'(1) = 1$
6. $(x-2)y'' + y' + (x-2)\tan(x)y = 0, \quad y(3) = 1, y'(3) = 2$

Find fundamental sets of solutions for the given differential equations and initial time t_0 :

1. $y'' + y' - 2y = 0, t_0 = 0$
2. $y'' + 5y' + 4y = 0, t_0 = 1$

Check that the following pairs of functions are solutions to the ODEs. Are they fundamental sets of solutions?

1. $y'' + 4y = 0, y_1(t) = \cos(2t), y_2(t) = \sin(2t)$
2. $y'' - 2y' + y = 0, y_1(t) = e^t, y_2(t) = te^t$
3. $x^2y'' - x(x+2)y' + (x+2)y = 0, (x > 0),\ y_1(x) = x, y_2(x) = xe^x$
4. $(1 - x\cot(x))y'' - xy' + y = 0, (0 < x < \pi),\ y_1(x) = x, y_2(x) = \sin(x)$

7 Complex numbers and complex roots of equations

Use Euler's formula to write following expressions in terms of sin and cos

1. $\exp(1 + 2i)$
2. $\exp(2 - 3i)$
3. $e^{i\pi}$
4. $e^{2-\frac{\pi}{2}i}$

In the following problems, find the general solution to the following differential equations:

1. $y'' - 2y' + 2y = 0$
2. $y'' - 2y' + 6y = 0$
3. $y'' + 2y' - 8y = 0$
4. $y'' + 2y' + 2y = 0$
5. $y'' + 6y' + 13y = 0$
6. $4y'' + 9y = 0$
7. $y'' + 2y' + 1.25y = 0$
8. $9y'' + 9y' - 4y = 0$
9. $y'' + y' + 1.25y = 0$
10. $y'' + 4y' + 6.25y = 0$

Find solutions to the following initial value problems:

1. $3u'' - u' + 2u = 0,\ u(0) = 2,\ u'(0) = 0.$
2. $5u'' + 2u' + 7u = 0,\ u(0) = 2,\ u'(0) = 1.$
3. $y'' + 2y' + 6y = 0,\ y(0) = 2,\ y'(0) = \alpha \geq 0.$
4. $y'' + 2ay' + (a^2 + 1)y = 0,\ y(0) = 1,\ y'(0) = 0.$

8 Reduction of order and repeated roots

For the following problems, find the general solution of the ODE:

1. $y'' - 2y' + y = 0$
2. $9y'' + 6y' + y = 0$
3. $4y'' - 4y' - 3y = 0$
4. $4y'' + 12y' + 9y = 0$
5. $y'' - 2y' + 10y = 0$
6. $y'' - 6y' + 9y = 0$
7. $4y'' + 17y' + 4y = 0$

8. $16y'' + 24y' + 9y = 0$
9. $25y'' - 20y' + 4y = 0$
10. $2y'' + 2y' + y = 0$

In the following, use reduction of order to find the second solution to the following ODEs

1. $t^2y'' - 4ty' + 6y = 0, \quad t > 0; \quad y_1(t) = t^2$
2. $t^2y'' + 2ty' - 2y = 0, \quad t > 0; \quad y_1(t) = t$
3. $t^2y'' + 3ty' + y = 0, \quad t > 0; \quad y_1(t) = t^{-1}$
4. $t^2y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0; \quad y_1(t) = t$
5. $xy'' - y' + 4x^3y = 0, \quad x > 0; \quad y_1(x) = \sin(x^2)$
6. $(x-1)y'' - xy' + y = 0, \quad x > 1; \quad y_1(x) = e^x$
7. $x^2y'' - (x - 0.1875)y = 0, \quad x > 0; \quad y_1(x) = x^{1/4}e^{2x^{1/2}}$
8. $x^2y'' + xy' + (x^2 - 0.25)y = 0, \quad x > 0; \quad y_1(x) = x^{-1/2}\sin(x)$

9 Second order equations: inhomogeneous equations

9.1 Undetermined Coefficients

Use the method of undetermined coefficients to find a particular solution of the problems, and then the general solution.

1. $y'' - 2y' - 3y = 3e^{2t}$
2. $y'' + 2y' + 5y = 3\sin(2t)$
3. $y'' - y' - 2y = -2t + 4t^2$
4. $y'' + y' - 6y = 12e^{3t} + 12e^{-2t}$
5. $y'' - 2y' - 3y = -3te^{-t}$
6. $y'' + 2y = 3 + 4\sin(2t)$
7. $y'' + 9y = t^2e^{3t} + 6$
8. $y'' + 2y' + y = 2e^{-t}$
9. $2y'' + 3y' + y = t^2 + 3\sin(t)$
10. $y'' + y = 3\sin(2t) + t\cos(2t)$
11. $u'' + \omega_0^2 u = \cos(\omega t), \omega^2 \neq \omega_0^2$
12. $u'' + \omega_0^2 u = \cos(\omega_0 t)$

9.2 Variation of parameters

Use the method of variation of parameters to determine the particular solution of the following problems.

1. $y'' - 5y' + 6y = 2e^t$
2. $y'' - y' - 2y = 2e^{-t}$
3. $y'' + 2y' + y = 3e^{-t}$
4. $4y'' - 4y' + y = 16e^{t/2}$
5. $y'' + y = \tan t, 0 < t < \pi/2$
6. $y'' + 9y = 9\sec^2(3t), 0 < t < \pi/6$

7. $y'' + 4y' + 4y = t^{-2}e^{-2t}, t > 0$
8. $y'' + 4y = 3\csc(2t), 0 < t < \pi/2$
9. $4y'' + y = 2\sec(t/2), -\pi < t < \pi$
10. $y'' - 2y' + y = e^{t}/(1 + t^{2})$