Ordinary Differential Equations Practice Solutions

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1 Introductory thoughts about ODEs

Determine the value of r for which the following ODEs have solutions of the form $y = e^{rt}$:

1. $r = -3$
2. $r = \pm 1$
3. $r = 2, -3$
4. $r = 0, 1, 3$

Determine the values of r for which the following ODEs have solutions of the form $y = t^r$ for $t > 0$

1. $r = -1, -2$
2. $r = 1, 4$

2 Linear equations and the integrating factor method

Find the solutions to the following ODEs and initial value problems

1. $y = ce^{-3t} + (t/3) - (1/9) + e^{-2t}$
2. $y = ce^{2t} + t^3 e^{2t}/3$
3. $y = ce^{-t} + 1 + t^2 e^{-t}/2$
4. $y = (c/t) + (3\cos(2t))/4t + (3\sin(2t))/2$
5. $y = ce^{3t} - 2e^t$
6. $y = (c - t\cos(t) + \sin(t))/t^2$
7. $y = t^2 e^{-t^2} + ce^{-t^2}$
8. $y = (\arctan t + c)/(1 + t^2)^2$
9. $y = -te^{-t} + ct$
10. $y = ce^{-t/2} + 3t - 6$
11. $y = ce^{-t} + \sin(2t) - 2\cos(2t)$
12. $y = ce^{-t/2} + 3t^2 - 12t + 24$
13. $y = 3e^t + 2(t - 1)e^{2t}$
14. $y = (t^2 - 1)e^{-3t}/2$
15. $y = (3t^4 - 4t^3 + 6t^2 + 1)/(12t^2)$
16. $y = \sin(t)/t^2$
17. $y = (t + 2)e^{4t}$
18. $y = t^{-2}(\frac{\pi^2}{4} - 1 - t\cos(t) + \sin(t))$
19. $y = -(1 + t)e^{-t}/t^4$, $t \neq 0$
20. $y = (t - 1 + 2e^{-t})/t$, $t \neq 0$

3 Separable ODEs

Solve the given ODEs

1. $3y^2 - 2x^3 = c,\ y \neq 0$
2. $y^{-1} + \cos(x) = c$ if $y \neq 0,$ also $y = 0$ everywhere.
3. $3y^2 - 2\ln|1 + x^3| = c,\ x \neq -1,\ y \neq 0$
4. $3y + y^2 - x^3 + x = c,\ y \neq -3/2$
5. $2\tan(2y) - 2x - \sin(2x) = c$ if $\cos(2y) \neq 0,$ also $y = \pm(2n+1)\pi/4,\ \forall n \in \mathbb{Z},$ everywhere
6. $y = \sin(\ln(x) + c)$ if $x \neq 0$ and $|y| < 1,$ also $y = \pm 1$
7. $y^2 - x^2 + 2(e^y - e^{-x}) = c,\ y + e^y \neq 0$
8. $y + y^3/3 - x^4/4 = c$
9. $y = 1/(x^2 - x - 6)$
10. $y = -\sqrt{2x - 2x^2 + 4}$
11. $y = (2(1-x)e^x - 1)^{1/2}$
12. $y = (3 - 2\sqrt{1 + x^2})^{-1/2}$
13. $y = -(2\ln(1 + x^2) + 4)^{1/2}$
14. $r = 2/(1 - 2\ln(\theta))$
15. $y = -1/2 + \frac{1}{2}\sqrt{4x^2 - 15}$

4 Exact equations

Determine whether the following ODEs are exact. If so, find the solution.

1. Not exact.
2. $x^2 + 3x + y^2 - 2y = c$
3. $x^3 - x^2y + 2x + 2y^3 + 3y = c$
4. $x^2y^2 + 2xy = c$
5. $ax^2 + 2bxy + cy^2 = k$
6. Not exact.
7. $e^x \sin(y) + 2y\cos(x) = c,$ and $y = 0$
8. Not exact.
9. $e^{xy}\cos(2x) + x^2 - 3y = c$
10. Not exact.
11. $y\ln(x) + 3x^2 - 2y = c$
12. $x^2 + y^2 = c$

5 Second order equations with constant coefficients: real roots

Solve the given ODEs and initial value problems

1. $y = c_1 e^t + c_2 e^{-4t}$
2. $y = c_1 e^{t/2} + c_2 e^t$
3. $y = c_1 e^{t/2} + c_2 e^{-t/3}$
4. $y = c_1 e^{-t} + c_2 e^{-2t}$
5. $y = c_1 + c_2 e^{-5t}$
6. $y = c_1 e^{4t/3} + c_2 e^{-4t/3}$
7. $y = c_1 e^{(9+3\sqrt{5})t/2} + c_2 e^{(9-3\sqrt{5})t/2}$
8. $y = c_1 e^{(1+\sqrt{3})t} + c_2 e^{(1-\sqrt{3})t}$
9. $y = e^t$
10. $y = 5e^{-t}/2 - e^{-3t}/2$
11. $y = (13 + 5\sqrt{13})e^{(-5+\sqrt{13})t/2}/26 + (13 - 5\sqrt{13})e^{(-5-\sqrt{13})t/2}/26$
12. $y = -1 - e^{-3t}$
13. $y = 12e^{t/3} - 8e^{t/2}$
14. $y = (2/\sqrt{33})e^{(-1+\sqrt{33})t/4} - (2/\sqrt{33})e^{(-1-\sqrt{33})t/4}$
15. $y = e^{-9(t-1)}/10 + 9e^{t-1}/10$
16. $y = 2e^{-(t+2)/2}$

6 Second order equations: Wronskian and fundamental solution sets

Find the Wronskian of the following function pairs

1. $-5e^{-t}$
2. 1
3. e^{-4t}
4. $x^2 e^x$
5. $-e^{2t}$
6. 0

Find the longest interval in which the given initial value problems have a unique, twice-differentiable solution.

1. $0 < t < \infty$
2. $-\infty < t < 1$
3. $0 < t)4$
4. $0 < t < \infty$
5. $0 < x < 3$
6. $2 < x < 3\pi/2$

Find fundamental sets of solutions for the given differential equations and initial time t_0 :

1. $y_1(t) = \frac{1}{3}e^{-2t} + \frac{2}{3}e^t$, $y_2(t) = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^t$
2. $y_1 = -\frac{1}{3}e^{-4(t-1)} + \frac{4}{3}e^{-(t-1)}$, $y_2 = y_1 = -\frac{1}{3}e^{-4(t-1)} + \frac{1}{3}e^{-(t-1)}$

Check that the following pairs of functions are solutions to the ODEs. Are they fundamental sets of solutions?

1. Yes.
2. Yes.
3. Yes.
4. Yes.

7 Complex numbers and complex roots of equations

Use Euler's formula to write following expressions in terms of sin and cos

1. $e \cos 2 + ie \sin 2$
2. $e^2 \cos 3 - ie^2 \sin 3$
3. -1
4. $e^2 \cos(\pi/2) - ie^2 \sin(\pi/2) = -e^2 i$

In the following problems, find the general solution to the following differential equations:

1. $y = c_1 e^t \cos(t) + c_2 e^t \sin(t)$
2. $y = c_1 e^t \cos(\sqrt{5}t) + c_2 e^t \sin(\sqrt{5}t)$
3. $y = c_1 e^{2t} + c_2 e^{-4t}$
4. $y = c_1 e^{-t} \cos(t) + c_2 e^{-t} \sin(t)$
5. $y = c_1 e^{-3t} \cos(2t) + c_2 e^{-3t} \sin(2t)$
6. $y = c_1 \cos(3t/2) + c_2 \sin(3t/2)$
7. $y = c_1 e^{-t} \cos(t/2) + c_2 e^{-t} \sin(t/2)$
8. $y = c_1 e^{t/3} + c_2 e^{-4t/3}$
9. $y = c_1 e^{-t/2} \cos(t) + c_2 e^{-t/2} \sin(t)$
10. $y = c_1 e^{-2t} \cos(3t/2) + c_2 e^{-2t} \sin(3t/2)$

Find solutions to the following initial value problems:

1. $u = 2e^{t/6} \cos(\sqrt{23}t/6) - (2/\sqrt{23})e^{t/6} \sin(\sqrt{23}t/6$
2. $u = 2e^{-t/5} \cos(\sqrt{34}t/5) + (7/\sqrt{34})e^{-t/5} \sin(\sqrt{34}t/5$
3. $y = 2e^{-t} \cos(\sqrt{5}t) + ((\alpha + 2)/\sqrt{5})e^{-t} \sin(\sqrt{5}t)$
4. $y = e^{-at} \cos(t) + ae^{-at} \sin(t)$

8 Reduction of order and repeated roots

For the following problems, find the general solution of the ODE:

1. $y = c_1 e^t + c_2 te^t$
2. $y = c_1 e^{-t/3} + c_2 te^{-t/3}$
3. $y = c_1 e^{-t/2} + c_2 e^{3t/2}$
4. $y = c_1 e^{-3t/2} + c_2 te^{-3t/2}$
5. $y = c_1 e^t \cos(3t) + c_2 e^t \sin(3t)$
6. $y = c_1 e^{3t} + c_2 te^{3t}$
7. $y = c_1 e^{-t/4} + c_2 e^{-4t}$

8. $y = c_1 e^{-3t/4} + c_2 t e^{-3t/4}$
9. $y = c_1 e^{2t/5} + c_2 t e^{2t/5}$
10. $y = e^{-t/2}\cos(t/2) + c_2 e^{-t/2}\sin(t/2)$

In the following, use reduction of order to find the second solution to the following ODEs

1. $y_2 = t^3$
2. $y_2 = t^{-2}$
3. $y_2 = t^{-1}\ln(t)$
4. $y_2 = te^t$
5. $y_2 = \cos(x^2)$
6. $y_2 = x$
7. $y_2 = x^{0.25} e^{-2\sqrt{x}}$
8. $y_2 = x^{-1/2}\cos(x)$

9 Second order equations: inhomogeneous equations

9.1 Undetermined Coefficients

Use the method of undetermined coefficients to find a particular solution of the problems, and then the general solution.

1. $y = c_1 e^{3t} + c_2 e^{-t} - 2e^{2t}$
2. $y = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + 3\sin(2t)/17 - 12\cos(2t)/17$
3. $y = c_1 e^{-t} + c_2 e^{2t} - 7/2 + 3t - 2t^2$
4. $y = c_1 e^{2t} + c_2 e^{-3t} + 2e^{3t} - 3e^{-2t}$
5. $y = c_1 e^{3t} + c_2 e^{-t} + 3te^{-t}/16 + 3t^2 e^{-t}/8$
6. $y = c_1 + c_2 e^{-2t} + 3t/2 - \sin(2t)/2 - \cos(2t)/2$
7. $y = c_1\cos(3t) + c_2\sin(3t) + e^{3t}(9t^2 - 6t + 1)/162 + 2/3$
8. $y = c_1 e^{-t} + c_2 te^{-t} + t^2 e^{-t}$
9. $y = c_1 e^{-t} + c_2 e^{-t/2} + t^2 - 6t + 14 - 3\sin(t)/10 - 9\cos(t)/10$
10. $y = c_1\cos(t) + c_2\sin(t) - t\cos(2t)/3 - 5\sin(2t)/9$
11. $u = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + (\omega_0^2 - \omega^2)^{-1}\cos(\omega t)$
12. $u = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + (1/2\omega_0)t\sin(\omega_0 t)$

9.2 Variation of parameters

Use the method of variation of parameters to determine the particular solution of the following problems.

1. $Y = e^t$
2. $Y = -(2/3)te^{-t}$
3. $Y = (3/2)t^2 e^{-t}$
4. $Y = 2t^2 e^{t/2}$
5. $Y = -\cos(t)\ln(\tan(t) + \sec(t))$
6. $Y = \sin(3t)\ln(\tan(3t) + \sec(3t)) - 1$

7. $Y = -e^{-2t} \ln(t)$
8. $Y = (3/4) \sin(2t) \ln(\sin(2t)) - 3t \cos(2t)/2$
9. $Y = t \sin(t/2) + 2(\ln(\cos(t/2)) \cos(t/2)$
10. $Y = -(1/2)e^t \ln(1 + t^2) + te^t \arctan(t)$