PDEs 10422884 – Homework 10

This homework does not need to be handed in.

1. Let $h > 0$. Solve the problem

$$\begin{aligned} &u_t - u_{xx} + hu = 0,\ 0 < x < \pi,\ t > 0 \\ &u(0,t) = 0,\ u(\pi,t) = 1,\ t \geq 0 \\ &u(x,0) = 0,\ 0 \leq x \leq \pi \end{aligned}$$

   Is the solution classical? (Recall from class what you need to check!)

2. Solve the problem

$$\begin{aligned} &u_t - u_{xx} = 2t + (9t+31)\sin(3x/2),\ 0 < x < \pi,\ t > 0 \\ &u(0,t) = t^2,\ u_x(\pi,t) = 1,\ t \geq 0 \\ &u(x,0) = x + 3\pi,\ 0 \leq x \leq \pi \end{aligned}$$

   Is the solution classical?

3. Let $u(x,t)$ be a solution of the problem

$$\begin{aligned} &u_t - u_{xx} = 0,\ Q_T = \{(x,t) \mid 0 < x < \pi,\ 0 < t \leq T\} \\ &u(0,t) = u(\pi,t) = 0,\ 0 \leq t \leq T \\ &u(x,0) = \sin^2(x),\ 0 \leq x \leq \pi \end{aligned}$$

   Use the maximum principle to prove that $0 \leq u(x,t) \leq e^{-t}\sin(x)$ in the rectangle Q_T.

4. Consider the equation

$$\begin{aligned} &u_t - u_{xx} = 0,\ 0 < x < 1, t > 0 \\ &u(0,t) = u(1,t) = 0 \\ &u(x,0) = 4x(1-x) \end{aligned}$$

   (a) Show that $0 < u(x,t) < 1$ for all $t > 0$ and $x \in (0,1)$. (b) Show that $u(x,t) = u(1-x,t)$ for all $t \geq 0$ and $0 \leq x \leq 1$. (c) Use the energy method to show that $\int_0^1 u^2 dx$ is a strictly decreasing function of t.

5. Show that the maximum principle is not true for the equation $u_t = xu_{xx}$ with a variable coefficient. First verify that $u(x,t) = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle $\{-x \leq x \leq 2,\ 0 \leq t \leq 1\}$.