PDEs 10422884 – Homework 3

This homework must be handed in prior to the tutorial on ***April 27th, 2017****. Questions marked with* $*$ *will be graded, and will go towards your grade on the homework. Unmarked questions will be checked for completion (or a reasonable attempt).*

*1. Given the PDE

$$xyuu_x + y^2uu_y = x^2 + y^2, \quad x > 0,\, y > 0$$

and the auxiliary condition $u(x, x) = \sqrt{x}$, establish whether there exists a unique solution, and if so, find it. If there are infinitely many solutions, find two different ones. Otherwise show that there exist no solutions to the Cauchy problem. *Hint: the characteristic equations were solved in class.*

2. Given the Cauchy problem

$$u_x + u_y = 1, \quad u(x, x) = x.$$

Check that this problem does not have a unique solution, and sketch the characteristics and the projection of the initial curve on the xy plane. Recall that additional solutions can be given by specifying new curves that are non-tangential to the characteristics. For example, other solutions to this Cauchy problem can be given by using the new auxiliary conditions:

(a) $u(x, 1) = x$ which yields $u(x, y) = x$.

(b) $u(1, x) = x$ which yields $u(x, y) = y$.

(c) $u(x, -x) = 0$ which yields $u(x, y) = (x + y)/2$.

Add the projections of each of these initial curves to your sketch above, and verify the given solutions. Then find two different solutions by specifying other auxiliary conditions (*hint: can you involve an arbitrary function?*).

3.* Find a function $u(x, y)$ that solves the Cauchy problem

$$xu_x - yu_y = u + xy, \quad u(x, x) = x^2, \quad 1 \le x \le 2.$$

(a) Check whether the transversality condition holds.

(b) Draw the projections on the xy plane of the initial curve and the characteristic curves emanating from the points $(1, 1, 1)$ and $(2, 2, 4)$.

(c) Is the solution well defined in the entire plane?