PDEs 10422884 – Homework 5

This homework must be handed in prior to the tutorial on May 25th, 2017.

*1. Given the Cauchy problem for the wave equation

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = 0 \\ &u(x,0) = 0 \\ &u_t(x,0) = \begin{cases} 1 & |x| < a \\ 0 & |x| \geq a \end{cases} \end{aligned}$$

describing the motion of a string, sketch the profile of the string (u as a function of x) at the times $t = a/2c$, a/c, $3a/2c$, and $2a/c$.

Hint: Calculate

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds = \frac{1}{2c} \times (\text{length of } (x-ct, x+ct) \cap (-a,a)).$$

For the first case, $u(x, a/2c) = \frac{1}{2c} \times$ *(length of* $(x - a/2, x + a/2) \cap (-a, a)$*), which takes different values depending on whether* $|x| < a/2$, $a/2 < x < 3a/2$ *and so on.*

*2. The midpoint of a piano string of tension T, density ρ and length L is hit by a hammer with head diameter $2a$. A flea is sitting at a distance $l/4$ from one end (assume $a < l/4$). How long does it take the disturbance to reach the flea? *Hint: formulate a wave equation. If the units of density* $[\rho] = kg/m$ *and the units of tension* $[T] = kg \cdot m/s^2$*, what combination of these two will give a speed with* $[c] = m/s$*?*

3. Use the parallelogram identity to solve the initial-boundary value problem

$$\begin{aligned} &u_{tt} - u_{xx} = 0, \quad 0 < x < \infty, t > 0 \\ &u(x,0) = \sin(x), x \geq 0 \\ &u_t(x,0) = \cos(x), x \geq 0 \\ &u(0,t) = \sin(t), t \geq 0 \end{aligned}$$

4. *(Optional)* Show the following invariance properties of the wave equation: (a) any translation $u(x - y, t)$ of a solution is also a solution (for fixed y). (b) any derivative, e.g. u_x, of a solution is also a solution. (c) The dilation $u(ax, at)$ of any solution is again a solution for any constant a.