PDEs 10422884 – Homework 8

This homework must be handed in prior to the tutorial on June 22nd, 2017.

1. Consider the problem $u_t = ku_{xx}$ for $0 < x < l$, with the boundary conditions $u(0,t) = U$, $u_x(l,t) = 0$, and the initial condition $u(x,0) = 0$, where U is a constant. Find the solution in series form. *Hint: consider* $u(x,t) - U$.

2. In class you learned that for a Sturm Liouville problem of the form $(pv')' + qv + \lambda rv = 0$, if $q \leq 0$ and if $puu'|_a^b \leq 0$ for all u that satisfy the given boundary conditions, then all eigenvalues are nonnegative. Show for the equation $X'' = -\lambda X$ that Dirichlet and Neumann problems have only nonnegative eigenvalues. Under what conditions does the Robin problem

$$X'(0) - a_0 X(0) = 0$$
$$X'(l) + a_l X(l) = 0$$

have only nonnegative eigenvalues?

3. For $\phi(x) = x^2$ on $0 \leq x \leq 1$, calculate the coefficients in the expansion (Fourier sine series)

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x).$$

If you are not sure why the terms appearing in the sin are $n\pi x$, also solve the Dirichlet problem $X'' = -\lambda X$, $X(0) = X(1) = 0$.

4. A rod has length $l = 1$ and constant $k = 1$. Its temperature satisfies the heat equation. Its left end is held at temperature 0, its right end at temperature 1. Initially (at $t = 0$) the temperature is given by

$$\phi(x) = \begin{cases} \frac{5x}{2} \text{ for } 0 < x < 2/3 \\ 3 - 2x \text{ for } 2/3 < x < 1 \end{cases}$$

Find the solution, including the coefficients. *Hint: first find the equilibrium solution* $U(x)$, *and then solve the heat equation with initial condition* $u(x,0) = \phi(x) - U(x)$