PDEs 10422884 – Homework 9

This homework must be handed in prior to the tutorial on June 29th, 2017.

1. (a) Consider the eigenvalue problem $X'' = -\lambda X$ on the interval $[0, l]$ with Robin boundary conditions:

$$\begin{aligned} X' + X &= 0 \text{ on } x = 0, \\ X' &= 0 \text{ on } x = l. \end{aligned}$$

Investigate the problem assuming $\lambda = -\gamma^2 < 0$. Is/are there one (or many) negative eigenvalue(s)? Explain your answer graphically!

(b) Assuming you are treating a wave or heat equation on $[0, l]$, such that the expansion of the solution is

$$u(x,t) = \sum_n T_n(t) X_n(x) \tag{1}$$

where $X_n(x)$ are the eigenfunctions determined by the boundary conditions, and

$$T_n(t) = \begin{cases} A_n e^{-\lambda_n k t} \text{ for the heat eqn.} \\ A_n \cos(\sqrt{\lambda_n} ct) + B_n \cos(\sqrt{\lambda_n} ct) \text{ for the wave eqn.} \end{cases}$$

write down the form of the expansion assuming the Robin boundary conditions given above, and arbitrary initial conditions (e.g. $u = \phi$ on $t = 0$ for the heat equation). Try to interpret the results.

2. Solve the forced wave equation

$$\begin{aligned} &u_{tt} = c^2 u_{xx} + g(x) \sin(\omega t),\ 0 < x < l \\ &u = 0 \text{ on } x = 0, x = l \\ &u = u_t = 0 \text{ on } t = 0. \end{aligned}$$

For which values of ω can resonance occur? (Consult your notes from ODEs last semester).

3. Solve via series expansion the heat equation $u_t = u_{xx}$ in $(0, 1)$ with $u_x(0, t) = 0$, $u(1, t) = 1$, and $u(x, 0) = x^2$. Compute the first two coefficients explicitly. What is the equilibrium state (term that does not tend to zero with large time?)

4. Write the Legendre equation

$$(1 - x^2) y'' - 2xy' + n(n+1) y = 0$$

in Sturm-Liouville form

$$(pv')' + qv = 0.$$

Do the same for the Bessel equation

$$r^2 w''(r) + r w'(r) + (r^2 - \nu^2) w(r) = 0.$$

Hint: try dividing by r *and transforming* $x = r/\lambda$.

5. (Optional mathematical background) Recall that we call a relation $\langle\,,\,\rangle : X \times X \to \mathbb{R}$ from some vector space X into the real numbers an inner product (or scalar product) if $\forall x,\, y,\, x_1,\, x_2 \in X,\, \forall \lambda \in \mathbb{R}:$
(a) $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$, (b) $\langle \lambda x, y\rangle = \lambda \langle x, y\rangle$, (c) $\langle x, y\rangle = \langle y, x\rangle$, (d) $\langle x, x\rangle \geq 0$, (e) $\langle x, x\rangle = 0 \Leftrightarrow x = 0$. Show, for the vector space of integrable functions on an interval $[a, b] \in \mathbb{R}$ that

$$\langle f, g\rangle = \int_a^b f(x) g(x) dx$$

satisfies these properties and defines an inner product on this space.