$\boxed{1}$ Consider the inhomogeneous problem for $h>0$

$$u_t - u_{xx} + hu = 0 \quad \text{in} \quad 0<x<\pi,\ t>0$$
$$u(0,t)=0,\ u(\pi,t)=1 \quad ,\ t\geq 0$$
$$u(x,0)=0 \quad \text{for } x\in[0,\pi]$$

Make a change of variables to get homogeneous B.C.

$w = x/\pi$, $\quad v = u - w$, then v satisfies

$$v_t - v_{xx} = -hv - h\frac{x}{\pi}$$
$$v(0,t) = v(\pi,t) = 0$$
$$v(x,0) = -x/\pi$$

Expand $v(x,t) = \sum a_n(t) \sin\left(\frac{n\pi x}{e}\right) = \sum_{n=1}^{\infty} a_n(t) \sin(nx)$

with $a_n = \frac{2}{\pi}\int_0^{\pi} v(x,t)\sin(nx)\,dx$

Assume v is a smooth solution, then

$$v_t - v_{xx} + hv = \sum_{n=1}^{\infty} (a_n'(t) + n^2 a_n(t) + h a_n)\sin(nx) \overset{!}{=} -h\frac{x}{\pi}$$

$\xrightarrow[\text{integrate}]{\text{multiply by } \sin(mx)}$ $$(a_m'(t) + (m^2+h)a_m)\cdot \underbrace{\int_0^{\pi}\sin^2(mx)\,dx}_{=\pi/2} = \frac{-h}{\pi}\int_0^{\pi} x\sin(mx)\,dx$$

$$\Longleftrightarrow a_m' + (m^2+h)a_m = \frac{-2h}{\pi^2}\left(\frac{-\pi}{m}\cos(m\pi)\right) = \begin{cases} \frac{2h}{\pi m} & m \text{ even} \\ \frac{-2h}{\pi m} & m \text{ odd} \end{cases}$$

Solve this ODE with the integrating factor

$\mu = e^{(m^2+h)t}$

$$a_m\cdot e^{(m^2+h)t} = \pm\frac{2h}{\pi m}\int e^{(m^2+h)t}\,dt + C$$

$$\longrightarrow a_m = \frac{\pm 2h}{\pi m}\cdot\frac{1}{m^2+h} + Ce^{-(m^2+h)t}$$

Now $a_n(0) = \frac{2}{\pi}\int_0^{\pi} v(x,0)\sin(nx)\,dx \overset{I.C.}{=} -\frac{2}{\pi^2}\int_0^{\pi} x\sin(nx)\,dx$

$$= \frac{2}{\pi^2}\cdot\left(\frac{\pi}{n}\right)(-1)^n$$

Thus

$$\frac{(-1)^m \cdot h \cdot 2}{\pi m (m^2+h)} + C \overset{!}{=} \frac{2}{m\pi}(-1)^m$$

$$\Rightarrow \quad C = \frac{2\cdot(-1)^m \left(h - (m^2+h)\right)}{m\pi (m^2+h)}$$

$$\Longrightarrow \quad v(x,t) = \sum_{m=1}^{\infty} \left(\frac{(-1)^m 2 \left(h + e^{-(m^2+h)t} (h-(m^2+h))\right)}{m\pi (m^2+h)} \right) \sin(mx)$$

This solution cannot be classical at least at $t=0$, since the series fails to converge to -1 at the right endpoint $x=\pi$.

$\boxed{2}$ Solve
$$u_t - u_{xx} = 2t + (9t+31)\sin\left(\frac{3x}{2}\right)$$
$$u(0,t) = t^2 \quad u_x(\pi,t) = 1$$
$$u(x,0) = x + 3\pi$$
Get homog. B.C. via $u(x,t) = v(x,t) + x + t^2$
$$v_t - v_{xx} = (9t+31)\sin\frac{3x}{2}$$
$$v(0,t) = 0$$
$$v_x(\pi,t) = 0$$
$$v(x,0) = 3\pi$$
What are the eigenfcts?

Consider the ODEs
$$X'' = -\lambda X$$
$$X(0) = 0$$
$$X'(\pi) = 0$$

$\boxed{\lambda = \mu^2 > 0:}$
$$X = A\cos\mu x + B\sin\mu x$$
$$X(0) = A \stackrel{!}{=} 0$$
$$X'(\pi) = B\mu\cos(\mu\pi) \stackrel{!}{=} 0 \implies \pi\mu = \left(n - \tfrac{1}{2}\right)\pi, \; n = 1,2,\dots$$

$\boxed{\lambda = 0:}$
$$X = Ax + B$$
$$X(0) = 0 \implies B = 0$$
$$X'(\pi) = A = 0. \quad \checkmark$$

Write
$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\left(\left(n - \tfrac{1}{2}\right)x\right)$$
$$a_n(t) = \frac{2}{\pi}\int_0^{\pi} v(x,t)\sin\left(\left(n - \tfrac{1}{2}\right)x\right)dx$$
Assume $v(x,t)$ a smooth solution:
$$v_t - v_{xx} = \sum_{n=1}^{\infty}\left(a_n'(t) + \left(n - \tfrac{1}{2}\right)^2 a_n(t)\right)\sin\left(n - \tfrac{1}{2}\right)x = (9t+31)\sin\left(\frac{3x}{2}\right)$$
Multiply both sides by $\sin\left(m - \frac{1}{2}\right)x$ and integrate:
$$a_n'(t) + \left(n - \tfrac{1}{2}\right)^2 a_n(t) = 0 \qquad (n \neq 2)$$
$$a_2'(t) + \frac{9}{4}a_2(t) = (9t+31) \qquad (n = 2)$$

$\boxed{n \neq 2:}$ $$a_n(t) = e^{-(n-\frac{1}{2})^2 t} + C$$

$\boxed{n=2:}$ Integrating factor $\mu = e^{9/4t}$

$$e^{9/4t} \cdot a_2 = \int (9t+31) e^{9/4t} dt + C$$

$$= \frac{31 \cdot 4}{9} e^{9/4} + \frac{9t \cdot 4}{9} e^{9/4t} - \frac{9 \cdot 4^2}{9^2} e^{9/4t} + C$$

$$\Rightarrow a_2 = \frac{4}{9}(31 + 9t - 4) + C e^{-9/4t}$$

$$a_n(0) = \frac{2}{\pi} \int_0^{\pi} 3\pi \sin(n-\tfrac{1}{2})x \, dx = -6\left[\cos(n-\tfrac{1}{2})\pi - 1\right] \cdot \frac{1}{(n-\frac{1}{2})}$$

$$= \frac{6}{(n-\frac{1}{2})} = C_n \quad (n \neq 2)$$

$$= \underbrace{\frac{6}{n-\frac{1}{2}}}_{} = \frac{4}{9}(27) + C_2 \quad (n=2)$$

$$4 = \frac{6}{3/2} = 4 \cdot 3 + C_2 \Rightarrow C_2 = -8$$

Which determines all coefficients.

The solution cannot be classical, since at $x=0, t=0$

$$v(0,0) = 0 \neq 3\pi = v(0,0)$$
$$v_x(\pi,0) = 1 \neq 0 = v_x(\pi,0) \quad ↯$$

3 Let u be a solution of

$$u_t - u_{xx} = 0 \qquad Q_T = \{0 < x < \pi\} \times \{0 < t \le T\}$$
$$u(0,t) = u(\pi,t) = 0 \qquad t \in [0,T]$$
$$u(x,0) = \sin^2(x) \qquad x \in [0,\pi]$$

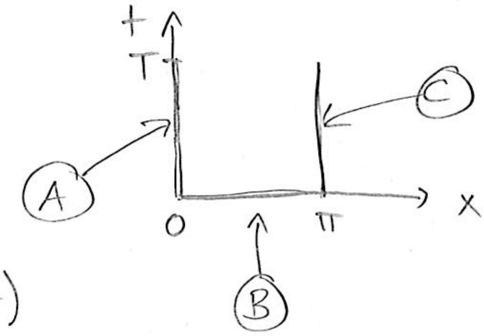
Use the maximum principle to prove $0 \le u(x,t) \le e^{-t}\sin(x)$ in the rectangle Q_T.

Employ the following trick: consider

$$(*) \begin{cases} v_t - v_{xx} = 0 & \text{on } Q_T \\ v(0,t) = v(\pi,t) = 0 & t \in [0,T] \\ v(x,0) = \sin(x) & x \in [0,\pi] \end{cases}$$

A solution to this problem $(*)$ is $v(x,t) = e^{-t}\sin(x)$

Consider the boundary $\partial_p Q_T$:



u is nonnegative on A, B, C by the B.C. & I.C. $\Rightarrow 0 \le u(x,t)$ (otherwise $\exists$ interior minimum)

Also: $u \le v = e^{-t}\sin(x)$ on A, B, C

(since $\sin^2(x) \le \sin(x)$ on $[0,\pi]$)

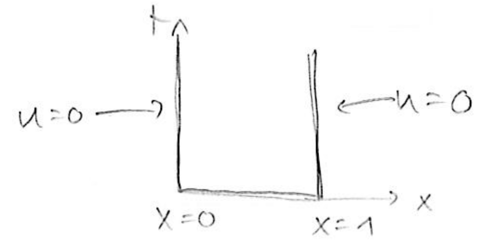
$\Rightarrow$ also $u \le e^{-t}\sin(x)$ in Q_T.

$\boxed{4}$ Consider $u_t = u_{xx}$ in $\{0<x<1,\ 0<t<\infty\}$

$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = 4x(1-x)$$

a) Show $0 < u(x,t) < 1 \quad \forall t>0,\ x\in(0,1)$

From the strong maximum principle the max/min must be obtained on the parabolic bdary



$$u(x,0) = 4x(1-x) = 4x - 4x^2$$

$$\frac{\partial u(x,0)}{\partial x} = 4 - 8x = 0 \quad \text{for } x = \tfrac{1}{2} \ ; \quad u(\tfrac{1}{2},0) = 2(\tfrac{1}{2}) = 1$$

$$\Longrightarrow \max u = 1, \quad \min u = 0$$

$$0 < u(x,t) < 1 \quad \text{in } (0,1)\times(0,\infty)$$

b) Show $u(x,t) = u(1-x,t) \quad \forall t \geq 0,\ x\in[0,1]$

$$u(x,0) = u(1-x,0) = 4(1-x)(1-(1-x)) = 4x(1-x)$$
$$u(1-0,t) = u(1,t) = 0$$
$$u(1-1,t) = u(0,t) = 0$$
$$u_t - u_{xx} = 0$$

So, since $u(x,t)$ and $u(1-x,t)$ satisfy the same initial-boundary-value problem, by uniqueness they must coincide.

c) $\int_0^1 u^2(x,t)\,dx \ : \quad \frac{d}{dt}\int_0^1 u^2(x,t)\,dx = 2\int_0^1 u u_t\,dx$

$$\overset{PDE}{=} 2\int_0^1 u\cdot u_{xx}\,dx \overset{P.I.}{=} 2\left[u u_x\Big|_0^1 - \int_0^1 (u_x)^2\,dx\right]$$

$$= 2\Big[\underbrace{u(1)}_{=0}u_x(1) - \underbrace{u(0)}_{=0}u_x(0) - \int_0^1 \underbrace{(u_x)^2\,dx}_{\geq 0}\Big] \leq 0.$$

$\boxed{5}$ Show the maximum principle does not hold for

$$u_t = x u_{xx}$$

a) $u = -2xt - x^2$: $u_t = -2x$

$$x u_{xx} = x \cdot (0 - 2) = -2x \ \Big\} \ ✓$$

$$\nabla u = \begin{pmatrix} -2x \\ -2t - 2x \end{pmatrix} = \vec{0} \quad \text{at} \quad (x,t) = (0,0)$$

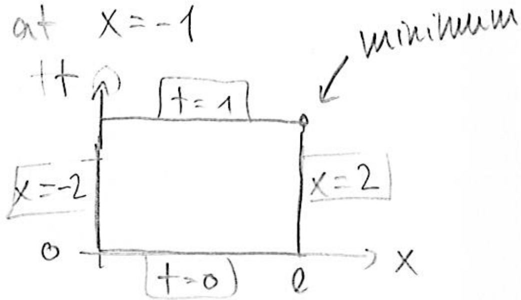
also check the sides: $t=0$, $t=1$, $x=-2$, $x=2$

$t=0$: $u = -x^2$: extremum at $x = 0$

$t=1$: $u = -2x - x^2$: extremum at $x = -1$

$x=-2$: $4t - 4$: $\frac{\partial}{\partial t} \neq 0$

$x = 2$: $-4t - 4$:



and corners, to find:

$u(0,0) = 0$

$u(-1,1) = 1$ ⟵ max. on top: $(x = -1, t = 1)$

$u(-2,0) = -4$

$u(2,1) = -8$ ⟵ minimum

b) At any maximum point on the top edge (i.e. on $t=1$) $u_t \geq 0$ (otherwise u increases as you move into the rectangle) and $u_{xx} \leq 0$ (otherwise u increases as you move along the edge).

If these inequalities are made strict, by introducing an auxiliary fct., then

$$\underbrace{u_t}_{\oplus} = \underbrace{u_{xx}}_{\ominus} \quad \text{is a contradiction.}$$

We have $u_t(-1,1) = 2 > 0$

$$x u_{xx}(-1,1) = 2 > 0$$

So $u_t = x u_{xx}$ gives no contradictions.