Homework 2

(*) $\boxed{1a}$ $yu_x - xu_y = 0 \quad (y>0)$

$\Leftrightarrow (y,-x)\cdot\nabla u = 0 \Leftrightarrow u$ constant in dir. $\vec{v} = (y,-x)$

i.e. along lines with slope $y' = -\frac{x}{y}$

$\Leftrightarrow yy' = -x \Leftrightarrow x^2+y^2 = C$

$\Rightarrow u = f(x^2+y^2)$

B.C. a) $u(x,0) = x^2 \Rightarrow f(x^2) = x^2 \Rightarrow f(w) = w$

$\Rightarrow u = x^2+y^2$

B.C. b) $u(x,0) = x \Rightarrow f(x^2) = x = f(w) = \sqrt{w}$

$\Rightarrow u = (x^2+y^2)^{1/2}$ provided $x,y \in \mathbb{R}$

(*) $\boxed{1b}$ Using characteristics, $yu_x - xu_y = 0$ has char. ODEs

$$\begin{cases} x_t = y \\ y_t = -x \\ u_t = 0 \end{cases}$$

$\Rightarrow x(t) = f_1(s)\sin(t) + f_2(s)\cos(t)$

$y(t) = f_1(s)\cos(t) - f_2(s)\sin(t)$

$u(t) = f_3(s)$

B.C. a) : $x_0(s) = f_2(s) = s$

$y_0(s) = f_1(s) = 0$

$u_0(s) = f_3(s) = s^2$

Hence: $\left.\begin{array}{l} x(t,s) = s\cos(t) \\ y(t,s) = -s\cdot\sin(t) \\ u(t,s) = s^2 \end{array}\right\} \Rightarrow x^2+y^2 = s^2$

$\Rightarrow u(x,y) = x^2+y^2.$

B.C. b) $x_0(s) = s$

$y_0(s) = 0$

$u_0(s) = s$

$\Rightarrow u(x,y) = \sqrt{x^2+y^2}.$

Homework 2

(*) $\boxed{2}$ $2u_t + 3u_x = 0$ with A.C. $u(x,0) = \sin(x)$

$\frac{du}{d\vec{v}} = 0$ for $\vec{v} = (2,3)$ or $u = c$ along $y = \frac{3}{2}x + C$

$\Rightarrow u(x,y) = f(2y - 3x)$

$u(0,y) = f(2y) = \sin(y). \Rightarrow f(w) = \sin\left(\frac{w}{2}\right)$

$\Rightarrow u(x,y) = \sin\left(\frac{2y-3x}{2}\right) \overset{\text{subst. } y=x,\ x=t}{=} \sin\left(x - \frac{3}{2}t\right)$

$\boxed{3}$ Solve $au_x + bu_y + cu = 0$ via $\xi = ax + by$, $\zeta = bx - ay$

$\frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx} + \frac{du}{d\zeta}\frac{d\zeta}{dx} = au_\xi + bu_\zeta$

$\frac{du}{dy} = bu_\xi - au_\zeta$

$\Rightarrow au_x + bu_y + cu = 0 \Leftrightarrow a^2u_\xi + abu_\zeta + b^2u_\xi - abu_\zeta + cu = 0$

$\Leftrightarrow (a^2+b^2)u_\xi + cu = 0$

$\Rightarrow u_\xi + \frac{c}{a^2+b^2}u = 0 \Rightarrow u = C(\zeta)\, e^{-\frac{c}{a^2+b^2}\xi}$

$\Rightarrow u(x,y) = F(bx - ay)\, e^{-\frac{c(ax+by)}{a^2+b^2}}.$

$\boxed{4}$ $u_x + u_y = 1$ is solved by $\xi = x + y$, $\zeta = x - y$.

$u_x = u_\xi + u_\zeta$, $u_y = u_\xi - u_\zeta$

$\Rightarrow u_x + u_y = 2u_\xi = 1 \Rightarrow u_\xi = 1/2 \Rightarrow u = \xi/2 + f(\zeta)$

$\Rightarrow u(x,y) = \frac{x+y}{2} + f(x-y).$

$\boxed{5}$ $u_t + x u_x = u^3$
$u(x,0) = \sin(x)$

Char. equations (now with indep. vas. called "s")

$$\begin{cases} t_s = 1 & \Longrightarrow t = s + f_1(\tau) \\ x_s = x & \Longrightarrow x = f_2(\tau) e^s \\ u_s = u^3 & \Longrightarrow u = \sqrt{\frac{1}{2} \frac{-1}{s + f_3(\tau)}} \end{cases}$$

and the initial curve $\Gamma(\tau) = \{ x_0 = \tau, t_0 = 0, u_0 = \sin(\tau) \}$

Hence

$$\left.\begin{aligned} f_1(\tau) &= 0 \\ f_2(\tau) &= \tau \\ f_3(\tau) &= \frac{-1}{2\sin^2(\tau)} \end{aligned}\right\} \quad \begin{aligned} t &= s \\ x &= \tau e^s \\ u &= \sqrt{\frac{1}{2} \frac{-1}{t - \frac{1}{2\sin^2(xe^{-t})}}} \end{aligned}$$

We observe: when $t = \frac{1}{2} \cdot \frac{1}{\sin^2(xe^{-t})}$

$\Longleftrightarrow \sin^2(xe^{-t}) = \frac{1}{2t}$ our solution blows up!