Homework 3

$\boxed{1}$ $xyuu_x + y^2uu_y = x^2 + y^2 \quad (x>0,\ y>0)$

$u(x,x) = \sqrt{x}$

first reformulate the PDE to

$$xu_x + yu_y = \frac{x^2+y^2}{yu}$$

and get characteristic equs

$$\left\{\begin{aligned} x_t &= x \\ y_t &= y \\ u_t &= \frac{x^2+y^2}{yu} \end{aligned}\right\} \xrightarrow{\text{gen. sol.}} u = \pm\sqrt{2e^t\left(\frac{c_1^2+c_2^2}{c_2}\right)+c_3}$$

Write the A.C. in parametric form:

$$\Gamma(s) = \{x_0(s) = s,\ y_0(s) = s,\ u_0(s) = \sqrt{s} \ /\ s>0\}$$

The coeff. of the equation are smooth about Γ, but

$$J|_{\Gamma} = \begin{vmatrix} a(s,s,\sqrt{s}) & b(s,s,\sqrt{s}) \\ x_0'(s) & y_0'(s) \end{vmatrix} = \begin{vmatrix} s & s \\ s' & s' \end{vmatrix} = 0$$

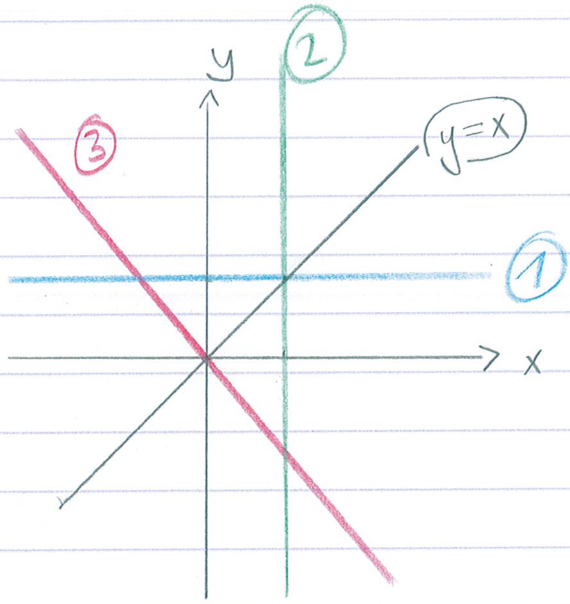
$\Rightarrow$ no unique solutions

Check: $$\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ s & s & c(s,s,\sqrt{s}) \\ 1 & 1 & u_0'(s) \end{vmatrix} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ s & s & 2\sqrt{s} \\ 1 & 1 & 1/2\sqrt{s} \end{vmatrix} \neq \vec{0}$$

$\Rightarrow \nexists$ solutions to the Cauchy problem

(this can also be seen by trying to solve directly.)

$\boxed{2}$ $u_x + u_y = 1$, $u(x,x) = x$



$$\begin{cases} x_t = 1 \implies x = t + f_1 \\ y_t = 1 \implies y = t + f_2 \\ u_t = 1 \implies u = t + f_3 \end{cases}$$

a) $u(x,1) = x$

$$\begin{cases} x = t+s \\ y = t+1 \\ u = t+s \end{cases} \implies u(x,y) = x \text{, a sol. to the Cauchy problem}$$

$\hookrightarrow u(x,x) = x$ ✓

b) $u(1,x) = x$

$$\begin{cases} x = t+1 \\ y = t+s \\ u = t+s \end{cases} \implies u(x,y) = y \text{, a sol. to the Cauchy problem}$$

$\hookrightarrow u(x,x) = x$ ✓

c) $u(x,-x) = 0$

$$\begin{cases} x = t+s \\ y = t-s \\ u = t \end{cases} \implies u(x,y) = \frac{x+y}{2} \text{, sol. to the Cauchy problem}$$

$\hookrightarrow u(x,x) = \frac{2x}{2} = x$ ✓

A general A.C. can be given by

$u(x,0) = f(x) \quad \forall f(x) : f(0) = 0$

$$\begin{cases} x = t+s \\ y = t \\ u = t + f(s) \end{cases} \implies u = y + f(x-y)$$

$\hookrightarrow u(x,x) = x + f(0) = x$ ✓

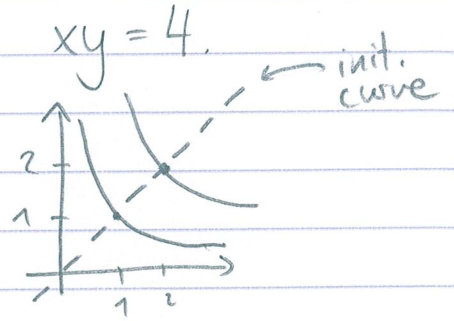
$\boxed{3}$ $xu_x - yu_y = u + xy$, $u(x,x) = x^2$, $1 \leq x \leq 2$

a) $J = \begin{vmatrix} s & -s \\ 1 & 1 \end{vmatrix} = 2s \neq 0$

b) The characteristics emanating from $(1,1,1)$ satisfy $xy = 1$.

Those from $(2,2,4)$ satisfy $xy = 4$.

Hence these look like



c) Solving the characteristic equations we see that

$$u(x,y) = xy\left(2\sqrt{\frac{x}{y}} - 1\right)$$

So we see that the sol. is <u>not</u> well defined in the entire plane.