Homework 4

[1] Find the regions in the xy-plane where

$$y u_{xx} - 2u_{xy} + x u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic

This equation is of general form with
$a_{11} = y, \; 2a_{12} = -2, \; a_{22} = x$

$$\Longrightarrow \quad a_{12}^2 = 1, \quad a_{11}a_{22} = xy$$

Hence the equation is parabolic if $xy = 1$
elliptic if $xy > 1$
hyperbolic if $xy < 1$

[2] Reduce the equ. $u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$
to $v_{xx} + v_{yy} + cv = 0$ by $u = ve^{\alpha x + \beta y}$ and $y' = \gamma y$.

Step 1: Let $u = ve^{\alpha x + \beta y}$. Calculate derivatives:

$$u_x = e^{\alpha x + \beta y}(\alpha v + v_x)$$
$$u_y = e^{\alpha x + \beta y}(\beta v + v_y)$$
$$u_{xx} = e^{\alpha x + \beta y}(\alpha^2 v + 2\alpha v_x + v_{xx})$$
$$u_{yy} = e^{\alpha x + \beta y}(\beta^2 v + 2\beta v_y + v_{yy})$$

Step 2: Substitute $u = ve^{\alpha x + \beta y}$ into the equation

$$e^{\alpha x + \beta y}\Big(v_{xx} + 3v_{yy} + (2\alpha - 2)v_x + (6\beta + 24)v_y + $$
$$+ (\alpha^2 + 3\beta^2 - 2\alpha + 24\beta + 5)v\Big) = 0.$$

Step 3: Choose α, β to eliminate derivatives of order 1: $\alpha = 1, \; \beta = -4$

Substituting these values, the equation becomes (dropping e^{x-4y})

$$v_{xx} + 3v_{yy} - 44v = 0$$

Step 4: We can clean this up by a change of scale in y: $y' = \gamma y \Rightarrow \partial/\partial y = \partial/\partial y' \cdot \gamma$

So, choosing $y' = \frac{1}{\sqrt{3}} \cdot y \quad v_{y'y'} = 3v_{yy}$

$$\Rightarrow \boxed{v_{xx} + v_{y'y'} - 44v = 0}$$

$\boxed{3}$ $u_{xx} + y u_{yy} = 0 \longrightarrow$ canonical form in the domain where the eq. is hyperbolic

Here $a_{11} = 1$, $2a_{12} = 0$
$a_{22} = y$ hence the equation is hyperbolic when $\underline{y < 0}$.

Recall char. eq. $\dfrac{dy}{dx} = \dfrac{a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} = \pm\sqrt{-y}$

$$\Rightarrow \int (-y)^{-1/2}\, dy = \pm x + C \Rightarrow 2 \cdot \sqrt{-y} \mp x = C$$

$$\Rightarrow \begin{array}{l} \varphi = 2\sqrt{-y} - x \\ \psi = 2\sqrt{-y} + x \end{array}$$ Simplify notation by $-y \mapsto \tilde{y}$

$$\xi = \varphi(x, \tilde{y}) = 2\tilde{y}^{1/2} - x, \quad \eta = \psi(x, \tilde{y}) = 2\tilde{y}^{1/2} + x$$

$\partial_x = -\partial_\xi + \partial_\eta$, $\quad \partial_x^2 = \partial_{\xi\xi} + \partial_{\eta\eta} - 2\partial_{\xi\eta}$

$\partial_{\tilde{y}} = \partial_\xi \cdot \tilde{y}^{-1/2} + \partial_\eta \tilde{y}^{-1/2}$

$$\partial_{\tilde{y}}^2 = \partial_{\xi\tilde{y}} \tilde{y}^{-1/2} + \partial_\xi \left(\frac{-\tilde{y}^{-3/2}}{2} \right) + \partial_{\eta\tilde{y}} \tilde{y}^{-1/2} + \partial_\eta \left(\frac{-\tilde{y}^{-3/2}}{2} \right)$$

$$= \partial_{\xi\xi} \tilde{y}^{-1} + \partial_{\xi\eta} \tilde{y}^{-1} + \partial_\xi \left(\frac{-\tilde{y}^{-3/2}}{2} \right) + \partial_{\eta\xi} \tilde{y}^{-1} + \partial_{\eta\eta} \tilde{y}^{-1} + \partial_\eta \left(\frac{-\tilde{y}^{-3/2}}{2} \right)$$

$$\partial_{\tilde{y}}^2 = \tilde{y}^{-1}\left(\partial_{\xi\xi} + 2\partial_{\xi\eta} + \partial_{\eta\eta} - \frac{\tilde{y}^{1/2}}{2}[\partial_\xi + \partial_\eta]\right)$$

Hence the PDE becomes

$$u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} - \tilde{y}\cdot\tilde{y}^{-1}\Big(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} - \frac{\tilde{y}^{-1/2}}{2}[u_\xi + u_\eta]\Big) = 0$$

$$\Leftrightarrow \boxed{-4u_{\xi\eta} - \frac{\tilde{y}^{-1/2}}{2}[u_\xi + u_\eta] = 0}$$

$\boxed{4}$ $xu_{xx} - yu_{yy} + \frac{1}{2}(u_x - u_y) = 0$

The equation is hyperbolic when $xy > 0$ and elliptic when $xy < 0$.

Characteristic equation: $\frac{dy}{dx} = \frac{\sqrt{xy}}{x}$

a) When $xy > 0$: two real roots $y' = \pm\sqrt{y/x}$, with e.g. $x > 0, y > 0$

$y^{1/2} \pm \cancel{x}^{1/2} = C$

so that

$$\xi = y^{1/2} + x^{1/2}$$
$$\eta = y^{1/2} - x^{1/2}$$

is a canonical transformation.

b) When $xy < 0$: two complex roots $y' = \pm i\sqrt{|y/x|}$

s.t. $2\cdot\text{sgn}(y)\cdot|y|^{1/2} = 2i\,\text{sgn}(x)\,|x|^{1/2} + C$

$\overset{\text{sgn}(x) = -\text{sgn}(y)}{\Longleftrightarrow}$ $|y|^{1/2} + i|x|^{1/2} = C$

$\xi = |y|^{1/2}$ and $\eta = |x|^{1/2}$ are a canonical transform.

$\boxed{5}$ $u_{tt} - 4u_{xt} - 5u_{xx} = 0$

First we notice that the PDE can be factored as two transport terms

(I) $(\partial_t - 5\partial_x)\underbrace{(\partial_t + \partial_x)u}_{=: v} = 0$

Hence we have a transport eqn.

$$v_t - 5v_x = 0$$

which has the gen. solution

$$v = g(x+5t)$$

This yields from (I)

$$u_t + u_x = g(x+5t)$$

which we can treat via the method of characteristics;

$$\frac{dt}{ds} = 1 \implies t = s + f_1$$

$$\frac{dx}{ds} = 1 \implies x = s + f_2$$

$$\frac{du}{ds} = g(x+5t) \implies u = G(x+5t) \text{ is a particular solution.}$$

Add to this a solution to the homogeneous problem

$u_t + u_x = 0$, say $u_h = f(x-t)$

and we have

$$u(x,t) = f(x-t) + G(x+5t)$$

Check:
$$\left.\begin{aligned} u_{tt} &= f'' + 25G'' \\ -4u_{xt} &= +4f'' - 20G'' \\ -5u_{xx} &= -5f'' - 5G'' \end{aligned}\right\} \Sigma = 0.$$