$\boxed{1}$ Given the Cauchy problem for the wave eqn

$$u_{tt} - c^2 u_{xx} = 0$$
$$u(x,0) = 0$$
$$u_t(x,0) = \begin{cases} 1 & |x| < a \\ 0 & |x| \geq a \end{cases}$$

<u>D'Alembert's formula:</u> $u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)\, ds = \frac{\text{length}\left((x-ct, x+ct) \cap (-a,a)\right)}{2c}$

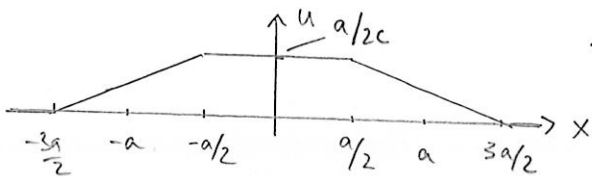
At $t = \frac{a}{2c}$: $u(x, \frac{a}{2c}) = \frac{1}{2c} \text{length}\left[\underbrace{(x - \frac{a}{2}, x + \frac{a}{2})}_{\text{length}=a} \cap \underbrace{(-a,a)}_{\text{length}=2a}\right]$

for $x \in (-a/2, a/2)$: $u = a/2c$

for $x \in (a/2, 3a/2)$: $u = \frac{-x}{2c} + \frac{3a}{4c}$

for $x \in (-3a/2, -a/2)$: $u = \frac{x}{2c} + \frac{3a}{4c}$

for $x > 3a/2$, $x < -3a/2$: $u = 0$

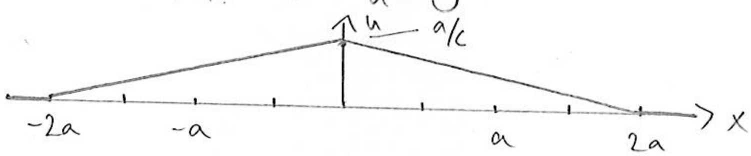


$t = a/2c$

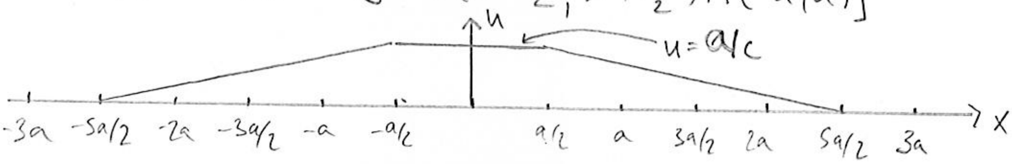
At $t = \frac{a}{c}$: $u(x, \frac{a}{c}) = \frac{1}{2c} \text{length}\left[(x-a, x+a) \cap (-a,a)\right]$

Has maximum at $x = 0$: $u = a/c$

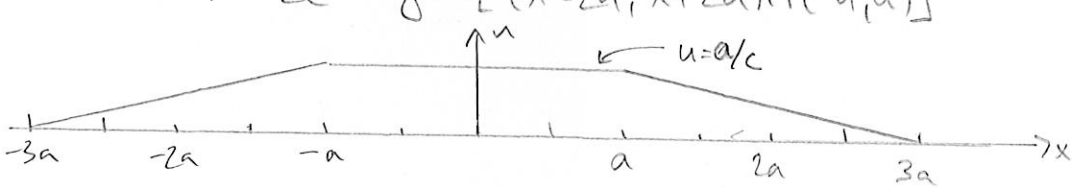
Vanishes for $|x| > 2a$: $u = 0$



At $t = 3a/2c$: $u(x, \frac{3a}{2c}) = \frac{1}{2c} \text{length}\left[(x - \frac{3a}{2}, x + \frac{3a}{2}) \cap (-a,a)\right]$



At $t = 2a/c$: $u(x, \frac{2a}{c}) = \frac{1}{2c} \text{length}\left[(x-2a, x+2a) \cap (-a,a)\right]$



2 A piano string of tension T, density ρ and length l is hit by a hammer of diameter $2a$.

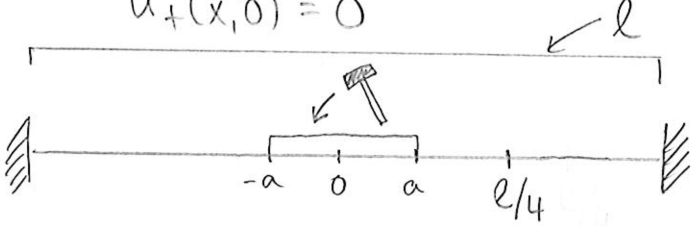
Since $[T] = \frac{kg\, m}{s^2}$, $[\rho] = \frac{kg}{m}$, $[c] = \frac{m}{s}$ $\Rightarrow c = \sqrt{T/\rho}$

and we have a wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

Hammer $\longrightarrow$ $u(x,0) = \begin{cases} 1 & |x| \leq a \\ 0 & |x| > a \end{cases}$

$$u_t(x,0) = 0$$



The disturbance must travel a distance $l/4 - a$ so the time this takes is

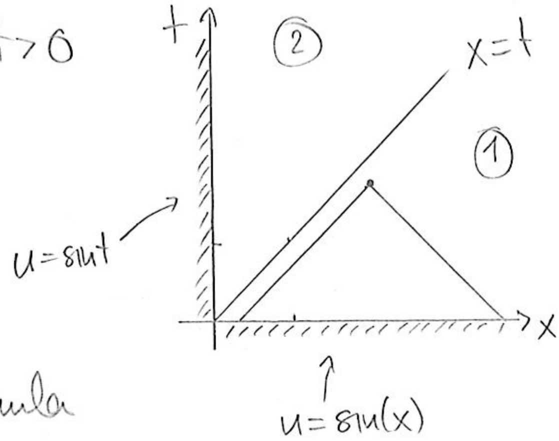
$$\left(\frac{l}{4} - a\right) \cdot \frac{1}{c} = \left(\frac{l}{4} - a\right) \cdot \sqrt{\rho/T} = t_0.$$

$\boxed{3}$ $\quad u_{tt} - u_{xx} = 0 \qquad 0 < x < \infty, \quad t > 0$

(i) $u(x,0) = \sin(x), \quad x \geq 0$

(ii) $u_t(x,0) = \cos(x), \quad x \geq 0$

(iii) $u(0,t) = \sin(t), \quad t \geq 0$



Region ①: use d'Alembert's formula

Region ②: use parallelogram identity

Note the compatibility conditions:

(i) $u(x,0) = \sin(x) \Rightarrow u(0,0) = 0$ ✓

(iii) $u(0,t) = \sin(t) \Rightarrow u(0,0) = 0$

(iii) $u(0,t) = \sin(t) \Rightarrow u_t(0,t) = \cos(t)$

$u_t(0,0) = 1$ ✓

(ii) $u_t(x,0) = \cos(x) \Rightarrow u_t(0,0) = 1$

Region ①: $u(x,t) = \frac{1}{2}\left(\sin(x-t) + \sin(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} \cos(y)\,dy$

Region ②: $u(x,t) = \sin(t-x) + \frac{1}{2}\left(\sin(x+t) - \sin(t-x)\right) + \frac{1}{2}\int_{t-x}^{t+x} \cos(y)\,dy$

Simplifying:

Region ①: $u(x,t) = \sin(x+t) \qquad x \geq t$

— " — ②: $u(x,t) = \sin(x+t) \qquad x < t$

$\Rightarrow u(x,t) = \sin(x+t)$ (plug in to check)