1 Solve $u_{tt} - c^2 u_{xx} = e^{ax}$

$u(x,0) = 0$

$u_t(x,0) = 0$

This is an inh. wave equation on the whole line, where we know the solution is given by a D'Alembert formula $+ \frac{1}{2c} \iint_C f$ for f the inhomogeneity.

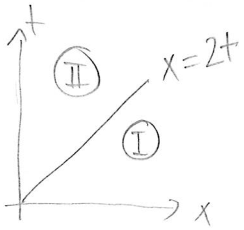
The d'Alembert terms all vanish

$$\Rightarrow 2c\,u(x,t) = \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^{ay}\, dy\, ds = \int_0^t \left[\frac{e^{ay}}{a} \right]_{x-c(t-s)}^{x+c(t-s)} ds$$

$$= \frac{1}{a} \int_0^t e^{a(x+c(t-s))} - e^{a(x-c(t-s))}\, ds = \frac{1}{a} \left[-\frac{e^{a(x+c(t-s))}}{ac} - \frac{e^{a(x-c(t-s))}}{ac} \right]_0^t$$

$$= \frac{1}{ca^2} \left[-e^{ax} - e^{ax} + e^{a(x+ct)} + e^{a(x-ct)} \right] = \frac{e^{a(x+ct)} + e^{a(x-ct)} - 2e^{ax}}{ca^2}$$

2 Solve $u_{tt} = 4u_{xx}$ on $0 < x < \infty$, $u(0,t) = 0$, $u(x,0) = 1$, $u_t(x,0) = 0$ using reflection.



In I $x > 2t$, the solution is given by d'Alembert's formula with $\phi(x) = 1$, $\psi(x) = 0$, $c = 2$:

$$u(x,t) = \frac{1}{2}(1+1) = 1.$$

In II: $x < 2t$ the initial data is extended via an odd reflection, so we have

$$u(x,t) = \frac{1}{2}(1-1) = 0$$

The singularity along the line $x = 2t$ is because the compatibility conditions fail:

$u(0,t) = 0 \Rightarrow u(0,0) = 0$

$u(x,0) = 1 \Rightarrow u(0,0) = 1$ ↯

$\boxed{3}$ Find a solution to the problem

$$u_{tt} - c^2 u_{xx} = 0 \;,\; x>0 \;,\; t>0$$
$$u(0,t)=0, \quad t>0$$
$$u(x,0)= xe^{-x}$$
$$u_t(x,0)=0 \quad x>0$$

For $x>ct$: use d'Alembert's formula to get

$$u(x,t) = \frac{1}{2}\left((x-ct)\,e^{-(x-ct)} + (x+ct)\,e^{-(x+ct)}\right)$$

For $0<x<ct$ we have

$$u(x,t) = \frac{1}{2}\left((x+ct)\,e^{-(x+ct)} - (ct-x)\,e^{-(ct-x)}\right)$$

Notice that the compatibility cond. hold.

$$u(x,0) = xe^{-x} \implies u(0,0)=0$$
$$u(0,t) = 0 \implies u(0,0)=0$$

and on $x=ct$ the solutions are continuous.

$\boxed{4}$

$$u_{tt} - u_{xx} = t^7 \qquad x\in\mathbb{R},\; t>0$$
$$u(x,0) = 2x+\sin(x) \qquad x\in\mathbb{R}$$
$$u_t(x,0)=0 \qquad x\in\mathbb{R}$$

A particular solution is $v=v(t)=\frac{1}{72}t^9$

Thus, solve the problem for $w=u-v$:

$$w_{tt} - w_{xx} = 0$$
$$w(x,0) = 2x+\sin(x)$$
$$w_t(x,0) = 0$$

$$\implies w(x,t) = \frac{1}{2}\left(2(x+t)+2(x-t)+\sin(x+t)+\sin(x-t)\right)$$
$$= 2x + \frac{1}{2}\left(\sin(x+t)+\sin(x-t)\right)$$

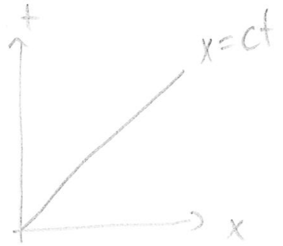
$$\implies u(x,t) = 2x + \frac{1}{2}\left(\sin(x+t)+\sin(x-t)\right) + \frac{t^9}{72}.$$

5 Solve $u_{tt} = c^2 u_{xx}$ $\quad 0<x<\infty,\ 0<t<\infty$

$u(x,0) = 0$

$u_t(x,0) = V$

$u_t(0,t) + a u_x(0,t) = 0$



first, consider the compatibility cond.:

$u(x,0)=0 \Rightarrow u_x(x,0)=0 \Rightarrow u_x(0,0)=0$

$u_t(x,0)=V \Rightarrow u_t(0,0)=V$

$u_t(0,t)+au_x(0,t)=0 \Rightarrow u_t(0,0)+au_x(0,0)=0$ $\lightning$

It is clear we will not get a continuous solution in the whole plane.

In $x>ct$: D'Alembert's formula holds as usual

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} V\, ds = \frac{V}{2c}\left[x+ct-x+ct\right] = Vt.$$

In $x<ct$ we have a Robin B.C.

One way to progress is to go back to the fundamentals - we know $u(x,t) = f(x+ct)+g(x-ct)$

$\Rightarrow u_t = cf'(x+ct) - cg'(x-ct)$

$\overset{\text{I.C.}}{\Rightarrow} u(x,0) = f(x)+g(x) = 0$

$\overset{\text{I.C.}}{\Rightarrow} u_t(x,0) = cf'(x) - cg'(x) = V$

$\Rightarrow f'+g'=0 \Rightarrow cf'+cf'=V \Rightarrow f' = \frac{V}{2c} \Rightarrow f(x) = \frac{Vx}{2c}$

simply because the I.C. must be satisfied.

Since $x+ct>0$, this formula for f holds in $0<x<\infty$, $0<t<\infty$

Using the B.C. $\quad 0 = u_t(0,t)+au_x(0,t) = f'(ct)(c+a) + g'(-ct)(c-a)$

$$\Rightarrow g'(-ct) = \frac{-(c+a)}{a-c} f'(ct)$$

or

$$g'(-y) = \frac{-(c+a)}{a-c} f'(y) \Rightarrow g(-y) = \frac{-(c+a)}{a-c} f(y) + C$$

$$= \frac{-(c+a)}{a-c}\frac{Vx}{2c} + C$$

For continuity at 0 set constant of integration $C=0$.
Now we have a formula for g for negative arguments, i.e. in $x<ct$

$$u(x,t)=f(x+ct)+g(x-ct)=\frac{V(x+ct)}{2c}+\left(\frac{-(c+a)}{a-c}\frac{V}{2c}(x-ct)\right)$$

$$=\frac{V}{2c}\left(\frac{1}{a-c}\Big((a-c)(x+ct)-(c+a)(x-ct)\Big)\right)$$

$$=\frac{V}{2c(a-c)}\left(\cancel{ax}-cx-\cancel{c^2t}+act-cx-\cancel{ax}+\cancel{c^2t}+act\right)$$

$$=\frac{V}{2c(a-c)}\Big(-2c(x-at)\Big)=\frac{V}{c-a}(x-at).$$

We notice that the solution is discontinuous at $\underline{x=ct}$.