Problem 1 - Homework 7

$$u_{tt} = c^2 u_{xx} - r u_t \qquad 0<x<l$$
$$u(t,0) = u(t,l) = 0$$
$$u(x,0) = \phi(x)$$
$$u_t(x,0) = \psi(x)$$

where $0<r<2\pi c/l$.

$$u(x,t) = X(x)T(t) \Rightarrow T''X = c^2 X''T - rXT'$$
$$\Leftrightarrow \frac{T''}{c^2T} + r\frac{T'}{c^2T} = \frac{X''}{X} = -\lambda$$

Hence we have the BVP
$$X'' = -\lambda X$$
$$X(0) = X(l) = 0$$
where we know $\sqrt{\lambda}\, l = n\pi \Rightarrow \lambda = \left(\frac{n\pi}{l}\right)^2$
So that
$$X_n(x) = \sin\left(\frac{n\pi}{l}x\right)$$

$$\begin{cases} T'' + rT' + \lambda c^2 T = 0 \\ T(0) = \phi(x) \\ T'(0) = \psi(x) \end{cases}$$

Is a problem for a homogeneous 2nd order ODE
Assume $T(t) = e^{at}$
$$T'' + rT' + \lambda c^2 T = 0 \Leftrightarrow e^{at}(a^2 + ra + \lambda c^2) = 0$$

Roots: $a = \dfrac{-r \pm \sqrt{r^2 - 4\underbrace{\lambda c^2}_{\frac{4n^2\pi^2c^2}{l^2}}}}{2}$

$$0 < r^2 < \frac{4\pi^2c^2}{l^2}$$

Since we have a Dirichlet problem, $\lambda \neq 0$

Since $r^2 < \lambda_1^2 < \lambda_2^2 < \dots \implies a = -r \pm i\sqrt{4\lambda_n c^2 - r^2}$

$$T(t) = e^{-rt}\left(A_n e^{i\sqrt{4\lambda_n c^2 - r^2}\,t} + B_n e^{-i\sqrt{4\lambda_n c^2 - r^2}\,t}\right)$$

$$\iff T(t) = e^{-rt}\left(\tilde{A}_n \cos\left(\sqrt{4\lambda_n c^2 - r^2}\,t\right) + \tilde{B}_n \sin\left(\sqrt{4\lambda_n c^2 - r^2}\,t\right)\right)$$

Hence $u(x,t) = \sum_{n=1}^{\infty} e^{-rt}\left(\tilde{A}_n \cos(\mu_n t) + \tilde{B}_n \sin(\mu_n t)\right) \sin\left(\frac{n\pi}{l}x\right)$

where $\mu_n = \sqrt{4c^2 \frac{n^2\pi^2}{l^2} - r^2}$

and $\tilde{A}_n = \frac{2}{l}\int_0^l \phi(x) \sin\left(\frac{n\pi}{l}x\right) dx$

$$\tilde{B}_n = \left(\frac{2}{l}\right)\left(-r \cdot \mu_n\right)\int_0^l \psi(x) \sin\left(\frac{n\pi}{l}x\right) dx$$

Since: $\int_0^l \sin^2\left(\frac{n\pi}{l}x\right) dx = \left[-\sin\left(\frac{n\pi}{l}x\right)\cos\left(\frac{n\pi}{l}x\right)\frac{l}{n\pi}\right]_0^l$

$$+ \int_0^l \cos^2\left(\frac{n\pi}{l}x\right) dx$$

$$= \left[-\sin(n\pi)\cos(n\pi) + 0\right] + \int_0^l 1 - \sin^2\left(\frac{n\pi}{l}x\right) dx$$

$$= l - \int_0^l \sin^2\left(\frac{n\pi}{l}x\right) dx$$

$$\implies 2\int_0^l \sin^2\left(\frac{n\pi}{l}x\right) dx = l$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n e^{\frac{1}{2}t(-r + i\sqrt{4\lambda_n c^2 - r^2})} + B_n e^{\frac{1}{2}t(-r - i\sqrt{4\lambda_n c^2 - r^2})} \right)$$

$$\cdot \sin\left(\frac{n\pi}{\ell} \cdot x\right)$$

$$= \sum_{n=1}^{\infty} e^{-rt/2} \left[A_n e^{it/2\sqrt{\mu}} + B_n e^{-it/2\sqrt{\mu}} \right] \sin\left(\frac{n\pi}{\ell} x\right)$$

$$= \sum_{n=1}^{\infty} e^{-rt/2} \left[\tilde{A}_n \cos\left(\frac{t}{2}\sqrt{\mu}\right) + \tilde{B}_n \sin\left(\frac{t}{2}\sqrt{\mu}\right) \right] \sin\left(\frac{n\pi}{\ell} x\right)$$

Problem 2 - Homework 7

$u_t = i u_{xx}$ on $(0, l)$, $u(t,0) = u(t,l) = 0$

$u(t,x) = X(x) T(t)$

$T' X = i T X'' \iff \frac{T'}{iT} = \frac{X''}{X} = -\lambda$

hence we have the BVP for X:

$$X'' = -\lambda X \quad \text{on} \quad 0 < x < l$$
$$X(0) = X(l) = 0$$

Thus $X(x) = A \sin(\sqrt{\lambda} x) + B \cos(\sqrt{\lambda} x)$

$X(0) = B = 0$

$X(l) = A \sin(\sqrt{\lambda} l) \stackrel{!}{=} 0$

$$\Longrightarrow \sqrt{\lambda} l = n\pi \Rightarrow \lambda = \left(\frac{n\pi}{l}\right)^2$$

Then the equ. for $T(t)$:

$$T' = -i \lambda T \iff T' = -i\left(\frac{n\pi}{l}\right)^2 T$$

$$\Longrightarrow T(t) = C e^{-i\left(\frac{n\pi}{l}\right)^2 t}$$

Hence

$$u(x,t) = X(x) T(t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{l} \cdot x\right) e^{-i\left(\frac{n\pi}{l}\right)^2 t}$$

where we know there are no negative or zero eigenvalues because we have a Dirichlet problem.

Problem 3 - Homework 7

$u_t = k u_{xx} \quad 0 < x < l$
$u(0,t) = u_x(l,t) = 0$

$$XT' = kX''T \Rightarrow \frac{T'}{kT} = \frac{X''}{X} = -\lambda$$

$\Rightarrow X(x) = A\cos$

Can we have $\lambda = 0$: $X(x) = ax + b$
$X(0) = 0 \Rightarrow b = 0$
$X'(l) = 0 \Rightarrow a = 0 \quad \Rightarrow \lambda \neq 0$

Can we have $\lambda < 0$: $X'' = \gamma^2 X \Rightarrow X(x) = A\cosh \gamma x + B\sinh \gamma x$
$X(0) = 0 \Rightarrow A = 0$
$X'(l) = 0 \Rightarrow B\gamma\cosh(\gamma l) = 0 \Rightarrow B = 0$
$\Rightarrow \lambda > 0.$

Hence we are left with positive eigenvalues $\lambda = \beta^2 > 0$

$$X(x) = A\cos(\beta x) + B\sin(\beta x)$$
$$X(0) = A = 0$$
$$X'(l) = \beta B\cos(\beta l) = 0 \Rightarrow \beta = \frac{(n+1/2)\pi}{l}$$

$$\Rightarrow \lambda_n = \beta^2 = \frac{(n+1/2)^2\pi^2}{l^2}$$

$$X_n(x) = \sin(\beta x) = \sin\left(\frac{(n+1/2)\pi x}{l}\right)$$

Finally $T' = -\lambda_n kT \Rightarrow T_n(t) = C_n e^{-k\lambda_n t}$
for each λ_n, $n = 0, 1, 2, \ldots$

$$\Rightarrow u(x,t) = \sum_{n=0}^{\infty} T_n(t) X_n(x) = \sum_{n=0}^{\infty} C_n \sin\left(\frac{(n+1)\pi x}{l}\right) e^{-k\frac{(n+\frac{1}{2})^2}{l^2}\pi^2 t}$$

Problem 4 - Homework 7

$$(x^2 v')' + \lambda v = 0 \quad , \quad v(1) = v(b) = 0$$

$$\hookrightarrow \; x^2 v'' + 2xv' + \lambda v = 0$$

This is an equation of Euler type. We can transform
$x = e^t \iff t = \ln(x)$
Then $d/dx = 1/x \; d/dt$ (recall from ODEs)
$d^2/dx^2 = -1/x^2 \; d/dt + 1/x^2 \; d^2/dt^2$

Then $x^2 v'' + 2xv' + \lambda v = 0 \iff v'' + v' + \lambda v = 0$

Make an Ansatz $v(t) = e^{at}$: $\cancel{a(a-1)}$

$$e^{at}(a^2 + a + \lambda) = 0 \implies a = -\frac{1}{2} \pm \frac{\sqrt{1-4\lambda}}{2}$$

Hence $v(t) = \left(A e^{\frac{1}{2}\sqrt{1-4\lambda}\,t} + B e^{-\frac{1}{2}\sqrt{1-4\lambda}\,t}\right) e^{-1/2 \cdot t}$

and using $t = \ln(x)$

$$v(x) = x^{-1/2}\left(A e^{\frac{1}{2}\sqrt{1-4\lambda}\ln x} + B e^{-\frac{1}{2}\sqrt{1-4\lambda}\ln x}\right)$$

$$v(1) = 0 \implies A = -B \implies v = x^{-1/2} A \left(e^{\frac{1}{2}\sqrt{1-4\lambda}\ln x} - e^{-\frac{1}{2}\sqrt{1-4\lambda}\ln x}\right)$$

$$v(b) = 0 \implies e^{\frac{1}{2}\sqrt{1-4\lambda}\ln x} = e^{-\frac{1}{2}\sqrt{1-4\lambda}\ln x}$$

Then either $\lambda = 1/4$, or if $\lambda > 1/4$, $\sqrt{1-4\lambda} \in \mathbb{C}$

$$\implies v(b) = b^{-1/2} A \left(e^{\frac{i}{2}\sqrt{4\lambda-1}\ln b} - e^{-\frac{i}{2}\sqrt{4\lambda-1}\ln b}\right)$$

$$= 2i b^{-1/2} A \sin\left(\frac{\sqrt{4\lambda-1}}{2}\ln b\right)$$

and $v(b) = 0 \iff \frac{\sqrt{4\lambda-1}}{2}\ln b = n\pi$

$$\iff \lambda = \frac{1}{4}\left(1 + \frac{4n^2\pi^2}{(\ln b)^2}\right) \quad \forall n = 0, 1, 2, \ldots$$

Note that for $\lambda = 1/4$, $v \equiv 0$, i.e. there are only trivial eigenfunctions. Hence start indexing at $n = 1$, and

eigenvalues: $\lambda_n = \frac{1}{4} + \frac{n^2\pi^2}{(\ln b)^2} \quad n = 1, 2$

eigenfunctions: $v_n = a_n x^{-1/2} \sin\left(\frac{\sqrt{4\lambda_n - 1}}{2} \ln x\right)$

$$= a_n x^{-1/2} \sin\left(\frac{n\pi}{\ln b} \ln x\right)$$

b) Solving $u_t = (x^2 u_x)_x \quad 1 < x < b, \ t > 0$

$u(1,t) = u(b,t) = 0, \ t \geq 0$

$u(x,0) = f(x) \quad 1 \leq x \leq b$

$$u = X(x)T(t) \implies T'X = (x^2 X'T)_x = 2xX'T + x^2 X''T$$

$$\iff \frac{T'}{T} = 2x\frac{X'}{X} + x^2\frac{X''}{X} = -\lambda \implies x^2X'' + 2xX' + \lambda X = 0$$

$$X(1) = X(b) = 0$$

$$T'(t) = -\lambda T \implies T = B_n e^{-\lambda_n t}$$

So $u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n t} x^{-1/2} \sin\left(\frac{n\pi}{\ln b} \ln x\right)$

with

$$B_n = \frac{\int_1^b f(x)\, x^{-1/2} \sin\left(\frac{n\pi}{\ln b} \ln x\right) dx}{\int_1^b x^{-1} \sin^2\left(\frac{n\pi}{\ln b} \ln x\right) dx}$$