Homework 8 - Problem 1

$$\left.\begin{array}{l} u_t = k u_{xx}, \quad 0 < x < l \\ u(0,t) = U \\ u_x(l,t) = 0 \\ u(x,0) = 0 \end{array}\right\} \xrightarrow{v = u - U:} \begin{array}{l} v_t = k v_{xx} \\ v(0,t) = 0 \\ v_x(l,t) = 0 \\ v(x,0) = -U \end{array}$$

The eigenfunctions for the homogeneous B.C. are

$$X_n = \sin\left((n+\tfrac{1}{2})\pi x/l\right), \quad \text{and} \quad \int_0^l X_n^2\, dx = \frac{l}{2}$$

Hence we know that the coefficients a_n in the eigenfunction expansion are

$$a_n = \frac{2}{l}\int_0^l (-U) X_n(x)\, dx = \frac{2}{l}\int_0^l -U \sin\left(\frac{(n+1/2)\pi x}{l}\right) dx$$

$$= -\frac{4U}{(2n+1)\pi}.$$

Hence we have the expansion for v

$$v(x,t) = \sum_{n=0}^{\infty} \frac{-4U}{(2n+1)\pi} \sin\left(\frac{(n+\frac{1}{2})\pi x}{l}\right) e^{-(n+\frac{1}{2})^2\pi^2 k t/l^2}$$

and for u:

$$u(x,t) = U - \sum_{n=0}^{\infty} \frac{-4U}{(2n+1)\pi} \sin\left(\frac{(n+\frac{1}{2})\pi x}{l}\right) e^{-(n+\frac{1}{2})^2\pi^2 k t/l^2}.$$

Homework 8 - Problem 2

For $X'' = -\lambda X$ we must verify that

$$X \cdot X' \Big|_a^b = X(b)X'(b) - X(a)X'(a) \leq 0$$

to ensure there are no negative eigenvalues.

If Dirichlet B.C. hold: $X(a) = X(b) = 0$

$$\Rightarrow XX' \Big|_a^b = 0$$

If Neumann B.C. hold: $X'(a) = X'(b) = 0$

$$\Rightarrow XX' \Big|_a^b = 0$$

If Robin B.C. hold: $X'(a) = a_0 X(a)$

$-X'(b) = a_\ell X(b)$

$$\Rightarrow XX' \Big|_a^b = -a_\ell X^2(b) - a_0 X^2(a)$$

This is non-positive if both a_0, a_ℓ are non-positive.

Homework 8 - Problem 3

The coefficients of the sine series are

$$A_n = 2\int_0^1 x^2 \sin(n\pi x)\,dx$$

$$= -\frac{2}{n\pi}\cos(n\pi x)x^2\Big|_0^1 + \frac{4}{n\pi}\int_0^1 x\cos(n\pi x)\,dx$$

$$= \frac{-2}{n\pi}\cos(n\pi) + \frac{4}{n^2\pi^2}\underbrace{\sin(nx\pi)\Big|_0^1}_{=0} - \frac{4}{n^2\pi^2}\int_0^1 \sin(n\pi x)\,dx$$

$$= \frac{-2}{n\pi}\cos(n\pi) + \frac{4}{n^3\pi^3}\left(\cos(n\pi)-1\right)$$

$$= \frac{-2}{n\pi}(-1)^n + \frac{4}{n^3\pi^3}\left((-1)^n-1\right)$$

So that $x^2 = \sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}(-1)^n + \frac{4}{n^3\pi^3}\left((-1)^n-1\right)\right)\sin(n\pi x)$

Note that this does not converge to 1 at $x=1$, which is clear when considering the incompatibility of the B.C. $X(1)=0$ with $\phi(1)=1$, or, for a heat eqn, e.g.

$$u_t = k u_{xx} \quad x\in(0,1),\ t>0$$
$$u(0,t)=u(1,t)=0$$
$$u(x,0)=x^2$$

Homework 6 - Problem 4

$$u_t = k u_{xx} \qquad 0<x<1, \quad t>0$$
$$u(0,t) = 0$$
$$u(1,t) = 1$$
$$u(x,0) = \phi(x) = \begin{cases} \frac{5x}{2} & 0<x<\frac{2}{3} \\ 3-2x & \frac{2}{3}<x<1 \end{cases}$$

We use $U(x) = x$ as an equilibrium solution, and solve

$$v_t = k v_{xx}$$
$$v(0,t) = 0$$
$$v(1,t) = 1$$
$$v(x,0) = \begin{cases} \frac{3x}{2} & 0<x<\frac{2}{3} \\ 3-3x & \frac{2}{3}<x<1 \end{cases}$$

The eigenfcts. are $X_n = \sin(n\pi x)$, $\langle X_n, X_n \rangle = \frac{1}{2}$

$$a_n = 2\int_0^1 (\phi(x) - x)\sin(n\pi x)\,dx$$

Recall that we have solved in class for the sine expansion of x and 1

$$1 = \sum_{m=1}^{\infty} \frac{2}{m\pi}(1-(-1)^m)\cdot \sin(m\pi x)$$
$$x = \sum_{m=1}^{\infty} \frac{2}{m\pi}(-1)^{m+1}\sin(m\pi x)$$

So: in $0<x<\frac{2}{3}$, $a_m = (-1)^{m+1}\frac{3}{m\pi}$

in $\frac{2}{3}<x<1$, $a_m = \frac{6}{m\pi}(1-(-1)^m) - \frac{6}{m\pi}(-1)^{m+1}$

$$\Rightarrow u(x,t) = x + \begin{cases} \sum_{m=1}^{\infty} (-1)^{m+1}\frac{3}{m\pi}\sin(m\pi x)e^{-(m\pi)^2kt} & 0<x<\frac{2}{3} \\ \sum_{m=1}^{\infty} \frac{6}{m\pi}(1-(-1)^m-(-1)^{m+1})\sin(m\pi x)e^{-(m\pi)^2kt} & \frac{2}{3}<x<1 \end{cases}$$

$\hookrightarrow$ Can be included in the series by writing $x = \sum \frac{2}{m\pi}(-1)^{m+1}\sin(m\pi x)$