Homework 9 - Problem 1:

a) Consider the Robin problem

$$X'' = -\lambda X \quad \text{on} \quad [0,\ell]$$
$$X' + X = 0 \quad \text{on} \quad x=0$$
$$X' = 0 \quad \text{on} \quad x=\ell$$

$\lambda = -\gamma^2 < 0$

Solving $X'' = \gamma^2 X$ yields a fundamental set of solutions $\{\cosh(\gamma x), \sinh(\gamma x)\}$
or $X = A\cosh(\gamma x) + B\sinh(\gamma x)$.

Hence $X'(0) + X(0) = A\gamma\sinh(\gamma\cdot 0) + B\gamma\cosh(\gamma\cdot 0) + A\cosh(0) + B\sinh(0)$

$= B\gamma + A = 0$

$X'(\ell) = A\gamma\sinh(\gamma\ell) + \underbrace{B\gamma}_{=-A}\cosh(\gamma\ell) = 0$

$\Rightarrow \tanh(\gamma\ell) = \frac{1}{\gamma}$

Thus: there is a negative eigenvalue if there is a solution to the equation

$\gamma\tanh(\gamma\ell) = 1$ for some γ.

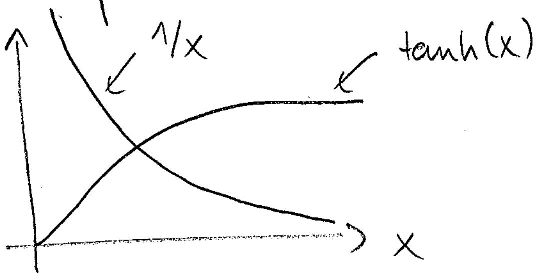
By the monotonicity of these curves ($1/\gamma$ is strictly decreasing from ∞ to 0 in $(0,\infty)$, $\tanh(\gamma\ell)$ increasing from 0 to 1 from $(0,\infty)$) there must be an intersection point.

There is <u>one</u> negative eigenvalue $\lambda = -\gamma^2$ with eigenvector

$$X(x) = \cosh(\gamma x) - \frac{1}{\gamma}\sinh(\gamma x).$$

[For specified l γ can be found numerically e.g. if $l=1$, then $\gamma = \pm 1.1997 \Rightarrow \lambda = -1.4392$]

The graphical interpretation of the monotonicity is



b Say we have the heat equ.

$$\begin{aligned} u_t &= k u_{xx} && \text{on } (0,l),\ t>0 \\ u_x + u &= 0 && \text{on } x=0 \\ u_x &= 0 && \text{on } x=l \\ u &= \phi && \text{on } t=0 \end{aligned}$$

We know that we have a usual collection of positive eigenvalues $\lambda_n = \beta_n^2 > 0$, and we have $\lambda_0 = -\gamma_0^2 < 0$ the one negative eigenvalue.

Check that $\lambda = 0$ is not an eigenvalue:

$$X'' = 0 \Rightarrow X = ax + b$$

$$\text{BC: } X' + X = a + ax + b\big|_{x=0} = 0$$

$$\Rightarrow a + b = 0$$

$$\text{BC: } X' = a\big|_{x=l} = 0 \Rightarrow a = 0$$

$$\Rightarrow b = 0 \Rightarrow X \equiv 0. \quad \checkmark$$

Hence we have : $\underbrace{\lambda_0}_{<0}, \underbrace{\lambda_1, \lambda_2, \lambda_3, \ldots}_{>0}$

Eigenfcts : $\boxed{\cosh \gamma_0 x - \frac{1}{\gamma_0}\sinh(\gamma_0 x)}$ $\boxed{\cos \beta_n x - \frac{1}{\beta_n}\sin(\beta_n x)}$

Where we recall that the eigenfcts. for the positive eigenvalues come from solving

$$X'' = -\beta^2 X$$
$$X' + X = 0 \quad \text{on } x = 0$$
$$X' = 0 \quad \text{on } x = \ell$$

$$\Rightarrow \quad X(x) = A\cos(\beta x) + B\sin(\beta x), \quad \text{etc.}$$

For the time-seperated part, we have

$$T' = -\lambda k T \quad \text{so} \quad \boxed{T_n(t) = A_n e^{-\lambda_n k t}}$$

Putting the solution together

$$u(x,t) = \sum_n X_n(x) T_n(t)$$

$$= A_0 e^{+\gamma_0^2 k t}\left(\cosh(\gamma_0 x) - \frac{1}{\gamma_0}\sinh(\gamma_0 x)\right)$$
$$+ \sum_{n=1}^{\infty} A_n e^{-\beta_n^2 k t}\left(\cos(\beta_n x) - \frac{1}{\beta_n}\sin(\beta_n x)\right)$$

We observe that, in general, the Robin boundary condition at $x=0$ supplies heat, while the Neumann condition at $x=\ell$ gives insulation. Hence the temperature grows exponentially due to $e^{\gamma_0^2 k t}$, unless $A_0 = 0$ by the initial data!

for a wave equation

$$u_{tt} = c^2 u_{xx}$$

the time-separated part of the problem is different, since

$$T'' = -\lambda c^2 T$$

so that

$$T_0'' = \gamma_0^2 c^2 T_0 \implies T_0(t) = A_0 e^{\gamma_0 ct} + B_0 e^{-\gamma_0 ct}$$

and $$T_n'' = -\beta_n^2 c^2 T_n \Rightarrow T_n(t) = A_n \cos(\beta_n ct) + B_n \sin(\beta_n ct)$$

However, the first term (corresponding to the negative eigenvalue λ_0)

$$(\underbrace{A_0 e^{\gamma_0 ct}}_{\text{exponential growth}} + \underbrace{B_0 e^{-\gamma_0 ct}}_{\text{exponential decay}})\left(\cosh(\gamma_0 x) - \frac{1}{\gamma_0}\sinh(\gamma_0 x)\right)$$

still presents problems unless A_0 is eliminated by the initial conditions.

Homework 9 - Problem 2:

Solve the forced wave equation

$$u_{tt} = c^2 u_{xx} + g(x)\sin(\omega t) \quad \text{on } (0,\ell)$$
$$u = 0 \quad \text{on} \quad x=0;\ x=\ell$$
$$u = u_t = 0 \quad \text{on } t=0.$$

Since we have a Dirichlet problem in x, write

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{\ell}\right)$$

Following the procedure used in the lectures, substitute directly in the PDE:

$$\sum_{n=1}^{\infty}\left[u_n'' + \frac{c^2 n^2\pi^2}{\ell^2} u_n\right]\sin\left(\frac{n\pi x}{\ell}\right) = g(x)\sin(\omega t)$$

Multiplying by $\sin\left(\frac{m\pi x}{\ell}\right)$ and integrating gives

$$\left[u_m'' + \frac{c^2 m^2\pi^2}{\ell^2} u_m\right]\cdot\frac{\ell}{2} = \int_0^{\ell} g(x)\sin(\omega t)\sin\left(\frac{m\pi x}{\ell}\right)dx$$

Call $\frac{n\pi}{\ell} =: \lambda_n$, then for any n

$$u_n''(t) + c^2\lambda_n^2 u_n(t) = \sin(\omega t)\cdot \underbrace{\left[\frac{2}{\ell}\int_0^{\ell} g(x)\sin(\lambda_n x)\,dx\right]}_{A_n}$$

Thus each mode satisfies the ODE for a forced, undamped harmonic oscillator

$$X'' + B^2 X = A\sin(\omega t)$$

This equation has as general solution

$$x(t) = \underbrace{\alpha \sin(Bt) + \beta \cos(Bt)}_{\text{sol. to hom. eq.}} + \underbrace{x_p(t)}_{\substack{\text{part. sol. to} \\ \text{inhom. eq.}}}$$

Recall from ODEs that we can find a particular solution by using the method of undetermined coefficients, i.e. assume $x(t) = a\sin(\omega t) + b\cos(\omega t)$ and plug in:

$$\left.\begin{aligned} x'' &= -a\omega^2 \sin(\omega t) - b\omega^2 \cos(\omega t) \\ B^2 x &= B^2 a \sin(\omega t) + B^2 b \cos(\omega t) \end{aligned}\right\} \Sigma \overset{!}{=} A\sin(\omega t)$$

$\Longrightarrow$ $b \equiv 0$ (since we don't want cos terms)

$$-a(\omega^2 - B^2) = A \quad \Longrightarrow \quad a = \frac{A}{B^2 - \omega^2}$$

for $B^2 \neq \omega^2$.

For u_n, this means

$$u_n(t) = \alpha_n \sin(c\lambda_n t) + \beta_n \cos(c\lambda_n t) + \frac{A_n}{c^2\lambda_n^2 - \omega^2} \sin(\omega t)$$

On the other hand, if $B^2 = \omega^2$, the method breaks down, and we recall that we need to introduce extra terms into our "guess":

$$x(t) = t\left(a\sin(\omega t) + b\cos(\omega t)\right)$$

Plug in:

$$x' = a\sin\omega t + b\cos\omega t + t(a\omega\cos\omega t - b\omega\sin(\omega t))$$

$$x'' = a\omega\cos\omega t - b\omega\sin\omega t$$
$$+ a\omega\cos\omega t - b\omega\sin(\omega t)$$
$$+ t(-a\omega^2\sin\omega t - b\omega^2\cos\omega t)$$

$$B^2 x = B^2 t(a\sin\omega t + b\cos\omega t)$$

$$x'' + B^2 x = 2a\omega\cos\omega t - 2b\omega\sin\omega t$$
$$- t\omega^2(a\sin\omega t + b\cos\omega t)$$
$$+ tB^2(a\sin\omega t + b\cos\omega t) = A\sin(\omega t)$$

Now: $\omega^2 = B^2 \Rightarrow 2a\omega\cos\omega t - 2b\omega\sin\omega t = A\sin(\omega t)$

$\Rightarrow a \equiv 0 \Rightarrow -2b\omega = A \Rightarrow b = -\dfrac{A}{2\omega}$

and $x(t) = -\dfrac{tA}{2\omega}\cos(\omega t)$

for u_n; if/where $c^2\lambda_n^2 = \omega^2$

$$u_n(t) = \alpha_n\sin(c\lambda_n t) + \beta_n\cos(c\lambda_n t) - \frac{tA_n}{2\omega}\cos(\omega t)$$

Homework 9 - Problem 3

$$u_t = u_{xx} \quad \text{in} \quad x \in (0,1),\ t>0$$
$$u_x(0,t) = 0$$
$$u(1,t) = 1$$
$$u(x,0) = x^2$$

We first shift the inhomogeneity away from the B.C. by $u(x,t) = 1 + v(x,t)$

$$v_t = v_{xx} \quad \text{in} \quad x \in (0,1),\ t>0$$
$$v_x(0,t) = 0$$
$$v(1,t) = 0$$
$$v(x,0) = x^2 - 1$$

Then we have, for the x-seperated problem

$$X'' = -\lambda X, \quad X'(0) = 0, \quad X(1) = 0$$

We recall that we may have positive and zero eigenvalues, but not negative (these need a Robin BC)

$\lambda = \mu^2 > 0$: $X'' = -\mu^2 X \Rightarrow X(x) = \alpha \cos(\mu x) + \beta \sin(\mu x)$

$X'(0) = \mu\beta = 0 \Rightarrow X(1) = \alpha \cos(\mu) = 0 \Rightarrow \mu = (n+\frac{1}{2})\pi$

$(n = 0,1,2,\dots)$

$\lambda = 0$: $X = \alpha x + \beta$. $\left.\begin{array}{l} X'(0) = \alpha = 0 \\ X(1) = \beta = 0 \end{array}\right\} \Rightarrow$ no negative evals

$$\boxed{\lambda_n = \left(n+\tfrac{1}{2}\right)^2 \pi^2, \quad X_n = \cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right)}$$

$$T' = -\lambda T \Rightarrow T_n = A_n e^{-(n+\frac{1}{2})^2\pi^2 t}$$

$$\Rightarrow u(x,t) = 1 + \sum_{n=0}^{\infty} A_n e^{-(n+\frac{1}{2})^2\pi^2 t} \cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right)$$

Now

$$u(x,0) = 1 + \sum_{n=0}^{\infty} A_n \cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right) \underset{\text{I.C.}}{\overset{!}{=}} x^2$$

Multiply by $\cos\left(\left(m+\frac{1}{2}\right)\pi x\right)$ and integrate from 0 to 1:

$$\underbrace{\int_0^1 \cos\left(\left(m+\tfrac{1}{2}\right)\pi x\right) dx}_{= \frac{\sin\left(\left(m+\frac{1}{2}\right)\pi\right)}{\left(m+\frac{1}{2}\right)\pi} = \frac{(-1)^m}{\left(m+\frac{1}{2}\right)\pi}} + A_m \int_0^1 \underbrace{\cos^2\left(\left(m+\tfrac{1}{2}\right)\pi x\right)}_{= \frac{1}{2} + \frac{1}{2}\cos\left((2m+1)\pi x\right)} dx = \int_0^1 x^2 \cos\left(m+\tfrac{1}{2}\right)\pi x \, dx$$

$$\Rightarrow \frac{(-1)^m}{\left(m+\frac{1}{2}\right)\pi} + \frac{A_m}{2} + \frac{A_m}{2}\Bigg(\overbrace{\frac{\sin\left((2m+1)\pi\right)}{(2m+1)\pi}}^{=0}\Bigg) = \int_0^1 x^2 \cos\left(m+\tfrac{1}{2}\right)\pi x \, dx$$

$$= \left[x^2 \frac{\sin \xi x}{\xi}\Big|_0^1 - \frac{2}{\xi}\int_0^1 x \sin(\xi x)\, dx\right]$$

$$= \left[\frac{\sin \xi}{\xi} - \frac{2}{\xi}\left[x\left(-\frac{\cos(\xi x)}{\xi}\right)\Bigg|_0^1 + \frac{1}{\xi}\int_0^1 \cos(\xi x)\, dx\right]\right]$$

$$= \left[\frac{\sin \xi}{\xi} - \frac{2}{\xi}\left[-\frac{\cos(\xi)}{\xi} + \frac{1}{\xi}\left[\frac{\sin x\xi}{\xi}\Big|_0^1\right]\right]\right]$$

$$= \left[\frac{\sin \xi}{\xi} - \frac{2}{\xi}\left[\underbrace{\frac{-\cos \xi}{\xi}}_{=0} + \frac{1}{\xi^2}\sin(\xi)\right]\right], \quad \xi = \left(m+\tfrac{1}{2}\right)\pi$$

$$\Downarrow$$

$$\sin(\xi) = (-1)^m$$

$$= \frac{(-1)^m}{\left(m+\frac{1}{2}\right)\pi} - \frac{2}{\left(m+\frac{1}{2}\right)^3\pi^3}(-1)^m$$

$$\cancel{\frac{(-1)^m}{\left(m+\frac{1}{2}\right)\pi}} + \frac{A_m}{2} = \cancel{\frac{(-1)^m}{\left(m+\frac{1}{2}\right)\pi}} - \frac{2}{\left(m+\frac{1}{2}\right)^3\pi^3}(-1)^m$$

$$\Longrightarrow \quad A_m = 4 \cdot \frac{(-1)^{m+1}}{\pi^3 (m+\frac{1}{2})^3}$$

The steady state can be found by letting $t \to \infty$:

All the terms $\quad e^{-(n+\frac{1}{2})^2 \pi^2 t} \longrightarrow 0$

and only the term 1 remains

$\rightarrow$ the steady state solution is $\underline{u=1}$, i.e. after a long time the initial condition $u = x^2$ "dies out", and the fact that the right end $u(1,t)$ is held at constant temperature 1, and there is no heat flux through the left end $(u_x(0,t)=0)$ means the temperature throughout the bar becomes $u=1$.

Homework 9 - Problem 4

Write in Sturm-Liouville form $(pv')' + qv = 0$

a) The Legendre eqn

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$p = (1-x^2), \quad q = n(n+1)$$

b) The Bessel eq.

$$r^2 w''(r) + r w'(r) + (r^2 - \nu^2) w(r) = 0$$

We first divide by $r \neq 0$

$$r w'' + w' + \left(r - \frac{\nu^2}{r}\right) w = 0$$

Now substitute $x = \frac{r}{\lambda}$, $\frac{d}{dr} = \frac{1}{\lambda}\frac{d}{dx}$

$$\frac{x}{\lambda} w'' + \frac{1}{\lambda} w' + \left(\lambda x + \frac{\nu^2}{\lambda x}\right) w = 0 \quad \Big| \cdot \lambda$$

$$x w'' + w' + \left(\lambda^2 x + \frac{\nu^2}{x}\right) w = 0$$

$$\Longrightarrow \quad \underset{\substack{\uparrow \\ p}}{(x w')'} + \underbrace{\left(\lambda^2 x + \frac{\nu^2}{x}\right)}_{\substack{\uparrow \\ q}} w = 0$$